Homework Assignment 2 - Solutions  Due Saturday 3/5/2022, uploaded to GradeScope.

Each question part is worth 1 point. There are 12 question parts. Assume that all categories are small. We define Fun(\mathcal{C}, \mathcal{D}) to be the category whose objects are functors \mathcal{C} \to \mathcal{D} and whose morphisms are natural transformations.

1. Suppose that \mathcal{F} and whose morphisms are natural transformations.

(a) Show that, for all objects \(x, y \in \text{Ob}\mathcal{C}\), the functor \(F\) provides a bijection

\[\text{Hom}_\mathcal{C}(x, y) \leftrightarrow \text{Hom}_\mathcal{D}(F(x), F(y))\]

that preserves composition, so that \(\text{End}_\mathcal{C}(x) \cong \text{End}_\mathcal{D}(F(x))\) as monoids.

(b) Show that \(x \cong y\) in \(\mathcal{C}\) if and only if \(F(x) \cong F(y)\) in \(\mathcal{D}\), so that \(F\) provides a bijection between the isomorphism classes of \(\mathcal{C}\), and of \(\mathcal{D}\).

(c) Let \(\mathcal{E}\) be a further category. Show that the functor categories \(\text{Fun}(\mathcal{C}, \mathcal{E})\) and \(\text{Fun}(\mathcal{D}, \mathcal{E})\) are naturally equivalent.

Solution. (a) For each morphism \(\alpha : x \to y\) in \(\mathcal{C}\) there is a morphism \(F(\alpha) : F(x) \to F(y)\), so \(F\) provides a mapping \(\text{Hom}_\mathcal{C}(x, y) \leftrightarrow \text{Hom}_\mathcal{D}(F(x), F(y))\), and this mapping preserves composition. Because \(F\) is an equivalence, there is another functor \(G : \mathcal{D} \to \mathcal{C}\) with natural isomorphisms \(\theta : FG \to 1_\mathcal{D}\) and \(\eta : GF \to 1_\mathcal{C}\). This means that for each \(\alpha : x \to y\) in \(\mathcal{C}\) we have \(\alpha = \eta_y(GF(\alpha))\eta_x^{-1}\) which shows that \(\alpha \mapsto F(\alpha)\) is one-to-one. Similarly the existence of \(\theta\) shows that \(\alpha \mapsto F(\alpha)\) is onto. Thus we have a bijection as claimed.

(b) If \(x \cong y\) in \(\mathcal{C}\) there are morphisms \(\alpha : x \to y\) and \(\beta : y \to x\) so that \(\beta \alpha = 1_x\) and \(\alpha \beta = 1_y\). Applying \(F\) these equations we get \(F(\beta)F(\alpha) = 1_{F(x)}\) and \(F(\alpha)F(\beta) = 1_{F(y)}\) so \(F(x) \cong F(y)\). Conversely, using the functor \(G\) from part (a), if \(F(x) \cong F(y)\) then \(GF(x) \cong GF(y)\) by the argument already used. Now \(\eta_z : GF(z) \to z\) is an isomorphism for all \(z\), so \(x \cong GF(x) \cong GF(y) \cong y\).

(c) With the previous notation, we get functors \(F^* : \text{Fun}(\mathcal{D}, \mathcal{E}) \to \text{Fun}(\mathcal{C}, \mathcal{E})\) and \(G^* : \text{Fun}(\mathcal{C}, \mathcal{E}) \to \text{Fun}(\mathcal{D}, \mathcal{E})\) given by precomposition with \(F\) and with \(G\), so that if \(\Phi : \mathcal{D} \to \mathcal{E}\) is a functor then \(F^*(\Phi) = \Phi \circ F\) and similarly if \(\Psi : \mathcal{C} \to \mathcal{E}\) is a functor then \(G^*(\Psi) = \Psi \circ G\).

We get a natural transformation \(\eta^* : F^*G^* = (GF)^* \to 1_{\text{Fun}(\mathcal{C}, \mathcal{E})}\) given at a functor \(\Psi\) by \(\eta^*_\Psi : (GF)^*(\Psi) = \Psi GF \to \Psi\) where \(\eta^*_\Psi(x) = \Psi(\eta_x)\). This is a natural isomorphism with inverse given by a similar construction applied to the inverse of \(\eta\). Also by a similar construction we get a natural isomorphism \(\theta^* : G^*F^* = (FG)^* \to 1_{\text{Fun}(\mathcal{D}, \mathcal{E})}\) given at a functor \(\Phi : \mathcal{D} \to \mathcal{E}\) by \(\theta^*_\Phi(z) = \Phi(\theta_z)\). This shows that the functor categories are naturally equivalent.

2. Let \(\mathcal{C}\) be a category and let \(x, y \in \text{Ob}\mathcal{C}\). Prove that if \(x \cong y\) then \(\text{Hom}_\mathcal{C}(x, -)\) and \(\text{Hom}_\mathcal{C}(y, -)\) are naturally isomorphic functors \(\mathcal{C} \to \text{Set}\).
Solution. Let $\alpha : x \rightarrow y$ and $\beta : y \rightarrow x$ be inverse isomorphisms. For each object $z$ in $C$ we have mappings $\alpha^*_z : \text{Hom}_C(y, z) \rightarrow \text{Hom}_C(x, z)$ and $\beta^*_z : \text{Hom}_C(x, z) \rightarrow \text{Hom}_C(y, z)$ given by $\alpha^*_z(f) = f\alpha$ and $\beta^*_z(g) = g\beta$. Now $\alpha^*$ and $\beta^*$ are natural transformations because if $u : z \rightarrow w$ then $u_*\alpha^*_z(f) = uf\alpha = \alpha^*_w(u*(f))$, and similarly with $\beta^*$. They are inverse to each other because, for each $z$, we have $\beta^*_z\alpha^*_z = 1_{\text{Hom}_C(y, z)}$ and $\alpha^*_z\beta^*_z = 1_{\text{Hom}_C(x, z)}$.

3. Let $F, G : C \rightarrow D$ be functors and $\eta : F \rightarrow G$ a natural transformation.

(a) Show that if, for all $x \in \text{ObC}$, the mapping $\eta_x : F(x) \rightarrow G(x)$ is an isomorphism in $D$, then $\eta$ is a natural isomorphism (meaning that it has a 2-sided inverse natural transformation $\theta : G \rightarrow F$).

(b) Suppose that $F$ is an equivalence of categories and that $F$ is naturally isomorphic to $G$, so $F \simeq G$. Show that $G$ is an equivalence of categories.

Solution. (a) If $\eta_x$ is an isomorphism for all $x$ we may define mappings $\theta_x := \eta^{-1}_x$. These $\theta_x$ define a natural transformation $\theta : G \rightarrow F$ because we know that for all morphisms $\alpha : x \rightarrow y$ we have $G(\alpha)\eta_x = \eta_yF(\alpha)$, so that $\theta_yG(\alpha) = \eta_y^{-1}G(\alpha)\eta_x\eta_x^{-1} = \eta_y^{-1}\eta_yF(\alpha)\eta_x^{-1} = F(\alpha)\theta_x$. It is inverse to $\eta$. (b) We will use the fact that if $F_1 : D \rightarrow E$ is another functor and $F \simeq G$ then $F_1F \simeq F_1G$. This is because the natural transformation $\eta : F \rightarrow G$ provides a natural transformation $F_1\eta : F_1F \rightarrow F_1G$ where, for each object $x$ of $C$, we have $(F_1\eta)_x = F_1(\eta_x) : F_1F(x) \rightarrow F_1G(x)$. If each $\eta_x$ is an isomorphism, so is $(F_1\eta)_x$, because functors take isomorphisms to isomorphisms. We will take $F_1 : D \rightarrow C$ to be an inverse equivalence to $F$, so that $F_1F \simeq 1_C$ and $FF_1 \simeq 1_D$. Now $F_1G \simeq F_1F \simeq 1_C$, and similarly $GF_1 \simeq FF_1 \simeq 1_D$. Finally we show that equivalence $\simeq$ is transitive. Suppose we have natural equivalences $\eta : F \rightarrow G$ and $\theta : G \rightarrow H$. Then $\theta\eta : F \rightarrow H$ is a natural transformation, where $(\theta\eta)_x := \theta_x\eta_x$ and if both $\theta_x$ and $\eta_x$ are isomorphisms, so is $(\theta\eta)_x$. It follows that $F_1G \simeq 1_C$ and $GF_1 \simeq 1_D$ so that $F_1$ is a natural inverse of $G$.

4. Let $G$ be a group, which we regard as a category $G$ with a single object, and with the elements of $G$ as morphisms. Let $F : G \rightarrow G$ be a functor.

(a) Show that $F$ is naturally isomorphic to the identity functor $1_G : G \rightarrow G$ if and only if the mapping $F : G \rightarrow G$, induced by $F$ on the set of morphisms, is an inner automorphism; that is, an automorphism of the form $c_g : G \rightarrow G$ for some $g \in G$, where $c_g(h) = ghg^{-1}$ for all $h \in G$.

(b) Show that self equivalences of $G$ are automorphisms of $G$.

(c) Show that the group of natural isomorphism classes of self equivalences of $G$ is isomorphic to $\text{Aut}(G)/\text{Inn}(G)$. (In the context of group theory, $\text{Inn}(G)$ denotes the set of inner automorphisms of $G$, and $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ is called the group of outer (or non-inner) automorphisms.)
Solution. (a) Writing the object of $G$ as $\ast$, there is a natural isomorphism $\theta : F \simeq 1$ if and only if there is an isomorphism $\theta_*$ in $G$ so that, for all morphisms $\alpha$ in $G$ the following diagram commutes:

$$
\begin{array}{ccc}
\ast & \xrightarrow{F(\alpha)} & \ast \\
\downarrow{\theta_*} & & \downarrow{\theta_*} \\
\ast & \xrightarrow{\alpha} & \ast
\end{array}
$$

The morphisms $\theta_*$, $\alpha$ and $F(\alpha)$ are all elements of $G$, and $F$ is a homomorphism $G \to G$. The diagram means that $F$ is $c_g$ where $g = \theta_*$. (b) If $F$ is a self equivalence of $G$ then $F(\ast) = \ast$ because $G$ has only one object, and $F$ is an isomorphism on the morphisms of $G$ by exercise 1(a).

(c) Each automorphism $F$ of $G$ provides a bijective map $f$ of the morphisms of $G$ to itself preserving composition, so an automorphism $f : G \to G$, and from part (b) we see that the correspondence $F \leftrightarrow f$ is an isomorphism $\text{Aut}(G) \cong \text{Aut}(G)$. We show that two automorphisms $F_1, F_2$ are equivalent if and only if the corresponding $f_1, f_2$ lie in the same coset of $\text{Inn}(G)$. Now $F_1 \simeq F_2$ if and only if $F_2^{-1}F_1 \simeq 1_G$ by an argument from question 3(b), which happens if and only if $f_2^{-1}f_1 \in \text{Inn}(G)$ by part (a) or, in other words, $f_1, f_2$ lie in the same coset of $\text{Inn}(G)$.

5. Let $I$ be the poset with two elements 0 and 1, and with $0 < 1$. If $P$ and $Q$ are posets we can regard them as categories $P$ and $Q$ whose objects are the elements of the posets, and where there is a unique morphism $x \to y$ if and only if $x \leq y$.

(a) Show that if $P$ and $Q$ are posets then a functor $P \to Q$ is ‘the same thing as’ an order-preserving map. (Don’t worry about any fancy interpretation of ‘the same thing as’!)

(b) Now consider two functors $F, G : P \to Q$, which we may regard as order-preserving maps $f, g : P \to Q$ by part (a). Show that the following three conditions are equivalent:

(i) there exists a natural transformation $F \to G$,

(ii) $f(x) \leq g(x)$ for all $x \in P$,

(iii) there is an order-preserving map $h : P \times I \to Q$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in P$. Here $P \times I$ denotes the product poset with order relation $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$, where $a_i \in P$ and $b_i \in I$.

Solution: (a) Let $F : P \to Q$ be a functor. If $x \leq y$ in $P$ we can regard this as a morphism $\alpha : x \to y$ in $P$, so that $F(\alpha) : F(x) \to F(y)$ is a morphism in $Q$, and $F(x) \leq F(y)$. Thus $F$ is an order preserving map. Conversely, given an order preserving map $f : P \to Q$ we obtain a functor $P \to Q$ that on objects is the same as $f$, and where if $x \to y$ is a morphism in $P$ we define the effect of the functor to be the unique morphism $f(x) \to f(y)$, which exists because $f(x) \leq f(y)$.

(b) Suppose (i) Then for each object $x$ of $P$ there is a morphism $\tau_x : F(x) \to G(x)$, so that $f(x) \leq g(x)$ for all $x \in P$. Thus (ii) holds.
Assuming (ii) holds, we show that the mapping defined in (iii) is order preserving. Suppose that \((a_1, b_1) \leq (a_2, b_2)\) and apply \(h\). If \(b_1 = b_2\) then \(h(a_1, b_1) \leq h(a_2, b_2)\) because either \(f\) or \(g\) is order preserving. The other possibility is \(b_1 = 0\) and \(b_2 = 1\), in which case this inequality holds because, in addition, \(f(x) \leq g(x)\) for all \(x\). Thus (iii) holds.

Assuming (iii) we define \(\tau_x\) to be the unique map \(f(x) \to g(x)\), which exists because \(f(x) = h(x, 0) \leq h(x, 1) = g(x)\). This shows that (i) holds.

6. Let \(1_{R-\text{mod}} : R-\text{mod} \to R-\text{mod}\) denote the identity functor. Let \(\text{Nat}(1_{R-\text{mod}}, 1_{R-\text{mod}})\) denote the set of natural transformations from this functor to itself, noting that this set has the structure of a ring (multiplication is composition and addition comes because we can add homomorphisms of \(R\)-modules, so that for two natural transformations \(\theta, \psi\) at an object \(x\) we have \((\theta + \psi)_x = \theta_x + \psi_x\). Show that \(\text{Nat}(1_{R-\text{mod}}, 1_{R-\text{mod}}) \cong Z(R)\).

Solution. We define mappings

\[
f : \text{Nat}(1_{R-\text{mod}}, 1_{R-\text{mod}}) \to R \quad \text{and} \quad g : Z(R) \to \text{Nat}(1_{R-\text{mod}}, 1_{R-\text{mod}})
\]
as follows. If \(\eta\) is such a natural transformation note that \(\eta_R : R \to R\) is an \(R\)-module homomorphism. We put \(f(\eta) = \eta_R(1_R) \in R\). If \(r \in Z(R)\) we define \(g(r)\) to be the natural transformation with \(g(r)_M : M \to M\) the mapping \(m \mapsto rm\). This is an \(R\)-module homomorphism because \(r\) lies in the center of \(R\), and \(g(r)\) is a natural transformation because if \(\alpha : M \to M\) is a homomorphism of \(R\)-modules then \(g(r)_M \alpha(m) = r\alpha(m) = \alpha(rm) = \alpha g(r)_M(m)\). We should verify several more things: \(f(\eta)\) lies in \(Z(R)\) and the two composite mappings \(fg\) and \(gf\) are the identity. If \(x\) is any element of \(R\) we have an \(R\)-module homomorphism \(\mu_x : R \to R\) where \(\mu_x(s) = sx\). Naturality of \(\eta\) means that \(\eta_R \mu_x = \mu_x \eta_R\). Applying these to \(1 \in R\) we get \(\eta_R(x) = \eta_R(1x) = x \eta_R(1) = \eta_R(1)x\), showing \(f(\eta)\) lies in \(Z(R)\). It is immediate that \(fg\) is the identity on \(Z(R)\). Finally we show that \(\eta = g(\eta_R(1_R))\) to see this consider the commutative diagram of \(R\)-modules

\[
\begin{array}{ccc}
R & \xrightarrow{\eta_R} & R \\
\downarrow & & \downarrow \\
M & \xrightarrow{\eta_M} & M
\end{array}
\]

where the vertical arrows are determined by \(1 \mapsto m\) for some arbitrary element \(m \in M\). Commutativity shows that \(\eta_M(m) = \eta_R(1_R)m\), which is what is needed.

**Extra question:** do not upload to Gradescope.

7. Let \(\mathcal{C}\) be a small category and let \(F, G : \mathcal{C} \to \text{Set}\) be functors. Show that a natural transformation of functors \(\tau : F \to G\) is an epimorphism in \(\text{Fun}(\mathcal{C}, \text{Set})\) if and only if for every object \(x\) of \(\mathcal{C}\), \(\tau_x : F(x) \to G(x)\) is a surjection; and it is a monomorphism if and only if for every object \(x\) of \(\mathcal{C}\), \(\tau_x : F(x) \to G(x)\) is a 1-1 map.
8. Write out a proof that if $G$ is the right adjoint of a functor $F$ with the property that $F$ preserves monomorphisms, then $G$ sends injective objects to injective objects.

9. Let $F : C \to D$ and $G : D \to C$ be functors with $F$ left adjoint to $G$, and with adjunction unit $\eta$ and counit $\epsilon$. Write out a proof that the second triangular identity holds, namely the following triangle commutes:

$$
\begin{array}{c}
G \\
\downarrow \eta_G \\
GFG \\
\downarrow G\epsilon \\
G
\end{array}
$$

10. Assume the axiom of choice in this question, or else make some assumption such as: everything is finite. Let $C$ be a category, and for each isomorphism class $\hat{x}$ of objects $x$, choose a fixed representative $u_{\hat{x}}$. For each object $x$ choose a fixed isomorphism $i_x : x \to u_{\hat{x}}$. Let $D$ be the full subcategory whose objects are the $u_{\hat{x}}$ where $x \in \text{Ob}C$. ‘Full’ means that for each pair of objects $y, z$ of $D$ we have $\text{Hom}_D(y, z) = \text{Hom}_C(y, z)$. Define $F(x) = \hat{x}$, and for each morphism $\alpha : x \to y$ define $F(\alpha) : F(x) \to F(y)$ to be $i_y \alpha i_x^{-1}$.

(a) Show that $F$ is a functor.

(b) Show that $F$ and the inclusion functor $\text{inc} : D \to C$ are inverse equivalences of categories $D \simeq C$. (It will help to assume that when $x = u_{\hat{x}}$, the chosen isomorphism is the identity $1_x$.)

(c) Deduce that the category $\text{Set}$ of finite sets is equivalent to the category with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$ and where $\text{Hom}(n, m)$ is the set of all mappings of sets from $n := \{1, \ldots, n\}$ to $m := \{1, \ldots, m\}$. We take $0 = \emptyset$.

(d) Deduce also the following: let $K$ be a field. Show that the category $\text{Vec}$ of finite dimensional vector spaces over $K$ is equivalent to the category $C$ with objects $\mathbb{N} := \{0, 1, 2, \ldots\}$, where $\text{Hom}_C(n, m)$ is the set $M_{m,n}(K)$ of $m \times n$ matrices with entries in $K$, and where composition of morphisms is matrix multiplication. In case $m$ or $n$ is zero, give a definition of $\text{Hom}_C(n, m)$ that will make this question make sense.

11. Let $C$ be a small category. A self-equivalence of $C$ is an equivalence of categories $F : C \to C$. Show that the set of natural isomorphism classes of self equivalences of $C$ is a group, with multiplication induced by composition of functors.