Math 8212 Commutative and Homological Algebra 2 Spring 2022

Homework Assignment 3 Due **Wednesday** 4/13/2022, uploaded to Gradescope. Each question part is worth 1 point. There are 17 question parts. You are on target for an A if you make a genuine attempt on at least half of them. We define $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to be the category whose objects are functors $\mathcal{C}\to\mathcal{D}$ and whose morphisms are natural transformations.

In these questions p is a prime. We will write an element $a_0+a_1p+a_2p^2+\cdots$ of the p-adic integers $\mathbb{Z}_p^\wedge$, where $0\le a_i\le p-1$, as a string $\cdots a_3a_2a_1a_0.$ with a point to the right of a_0.

1. a. Calculate the 3-adic expansion of $\frac{1}{2}$ in $\mathbb{Z}_3^\wedge$.

b. What fraction does the recurring 3-adic integer $\cdots\overline{0121}01211.$ represent?

c. Show that a p-adic integer is a negative (rational) integer if and only if it is of the form $\overline{(p-1)}a_n\cdots a_3a_2a_1a_0.$

d. Show that the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at (p) is the subset of $\mathbb{Z}_p^\wedge$ consisting of strings

$$\overline{a_m\cdots a_n}\cdots a_3a_2a_1a_0.$$

that eventually recur to the left.

2. In this question consider the 10-adic topology on $\mathbb{Z}$, determined by the powers of the ideal (10), with completion the 10-adic integers $\mathbb{Z}_{(10)}^\wedge$, and also the 2-adic topology on $\mathbb{Z}$ with completion $\mathbb{Z}_{(2)}^\wedge$

a. Show that a sequence of integers that is a Cauchy sequence in the 10-adic topology is also a Cauchy sequence in the 2-adic topology.

b. Show that the identity map $1:\mathbb{Z}\to\mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}_{(10)}^\wedge\to\mathbb{Z}_{(2)}^\wedge$.

c. Determine whether the identity map $1:\mathbb{Z}\to\mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}_{(2)}^\wedge\to\mathbb{Z}_{(10)}^\wedge$.

d. Using the fact that $\mathbb{Z}/10\mathbb{Z}\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/5\mathbb{Z}$ as a product of rings, show that $\mathbb{Z}_{(10)}^\wedge\cong A\times B$ for certain rings A,B that are also ideals of $\mathbb{Z}_{(10)}^\wedge$, with $A/(A\cap(10))\cong\mathbb{Z}/2\mathbb{Z}$ and $B/(B\cap(10))\cong\mathbb{Z}/5\mathbb{Z}$.

e. Show that $\mathbb{Z}_{(10)}^\wedge$ has just two maximal ideals, generated by 2 and 5.

f. Show that the composite morphism specified as the inclusion of the ideal $A\hookrightarrow\mathbb{Z}_{(10)}^\wedge$, followed by the ring homomorphism $\mathbb{Z}_{(10)}^\wedge\to\mathbb{Z}_{(2)}$ of part b, is surjective. (Consider using Nakayama's lemma.)

3. Find how many cube roots each of the following numbers has in $\mathbb{Z}_{(7)}^\wedge$: 1, 9, -4, 4, 12, 6. Also find how many cube roots each of the following numbers has in $\mathbb{Z}_{(5)}^\wedge$: 1, 2, 3, 4, 5.

4. Let I be an ideal of R. Consider the polynomial $f(x) = 3x^4 + x^2 + 5$ as a function $R \to R$. Show that f is continuous in the I-adic topology on R. (The I-adic topology on R is given by the distance function determined by the powers of I.)

5. For a category $\mathcal{C}$ and commutative ring R we may take the R-linear category $R\mathcal{C}$ to have the same objects as $\mathcal{C}$, and with $\mathrm{Hom}_{R\mathcal{C}}(x, y) = R\,\mathrm{Hom}_{\mathcal{C}}(x, y)$, the set of formal linear combinations of morphism $x \to y$ in $\mathcal{C}$. Composition is R-bilinear. The constant functor $\underline{R} : R\mathcal{C} \to R$-mod is the functor that assigns R to each object of $\mathcal{C}$, and the identity map 1_R to each morphism of $\mathcal{C}$.

a. Let $\mathcal{C}$ be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ with three objects, and five morphisms that are the two morphisms shown and the three identity morphisms for the objects. Show that the constant functor on $\mathcal{C}$ is representable as a linear functor $R\mathcal{C} \to R$-mod.

b. Let $\mathcal{D}$ be the category $\bullet \rightarrow \bullet \leftarrow \bullet$ with three objects, and five morphisms, with the two non-identity morphisms pointing in the opposite direction to the last example. Show that the constant functor is not representable.

c. Show that the inverse limit functor $\varprojlim : \mathrm{Fun}(\mathcal{D}, R\text{-mod}) \to R$-mod is representable, represented by the constant functor.

6. Let $\mathrm{Fun}(\mathcal{C}, \mathrm{abgps})$ be the category of functors from $\mathcal{C}$ to abelian groups, with natural transformations as morphisms. We may take as a definition that a sequence $F_1 \to F_2 \to F_3$ in $\mathrm{Fun}(\mathcal{C}, \mathrm{abgps})$ is exact if and only if, for all objects X in $\mathcal{C}$, the sequence of abelian groups $F_1(X) \to F_2(X) \to F_3(X)$ is exact. This is equivalent to other possible definitions of exactness. We may regard the inverse limit construction as a functor $\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{abgps}) \to$ abgps.

a. Let $\mathcal{C}$ be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ with three objects, and five morphisms that are the two morphisms shown and the three identity morphisms for the objects. Show that the functor $\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{abgps}) \to \mathrm{abgps}$ is exact.

b. Let $\mathcal{D}$ be the category $\bullet \rightarrow \bullet \leftarrow \bullet$ with three objects, and five morphisms, with the two non-identity morphisms pointing in the opposite direction to the last example. Show (by example, or by giving a reason) that the functor $\varprojlim : \mathrm{Fun}(\mathcal{D}, \mathrm{abgps}) \to \mathrm{abgps}$ is not exact in general.