Homework Assignment 3 - Solutions Due Sunday 4/17/2022, uploaded to Gradescope.

Each question part is worth 1 point. There are 17 question parts. You are on target for an A if you make a genuine attempt on at least half of them. We define \( \text{Fun}(C, D) \) to be the category whose objects are functors \( C \to D \) and whose morphisms are natural transformations.

In these questions \( p \) is a prime. We will write an element \( a_0 + a_1p + a_2p^2 + \cdots \) of the \( p \)-adic integers \( \mathbb{Z}^\wedge_p \), where \( 0 \leq a_i \leq p-1 \), as a string \( \cdots a_3a_2a_1a_0 \). with a point to the right of \( a_0 \).

1. a. Calculate the 3-adic expansion of \( \frac{1}{2} \) in \( \mathbb{Z}^\wedge_3 \).

   b. What fraction does the recurring 3-adic integer \( \cdots 012101211 \). represent?

   c. Show that a \( p \)-adic integer is a negative (rational) integer if and only if it is of the form \( \frac{(p-1)a_n \cdots a_3a_2a_1a_0}{(p-1)^n} \).

   d. Show that the localization \( \mathbb{Z}(p) \) of \( \mathbb{Z} \) at \( (p) \) is the subset of \( \mathbb{Z}^\wedge_p \) consisting of strings \( \bar{a}_m \cdots \bar{a}_n \cdots \bar{a}_3\bar{a}_2\bar{a}_1\bar{a}_0 \) that eventually recur to the left.

Solution: a. The multiplication sum

\[
\begin{array}{cccc}
\cdots & 1 & 1 & 1 & 2 \\
\times & 2 & \hline \\
\end{array}
\]

shows that \( \cdots 12 \). multiplied by 2 equals 1, so \( \cdots \bar{1}2 = \frac{1}{2} \).

b. Let \( x = \cdots \bar{0} \bar{1} \bar{2} \bar{1} \bar{0} \bar{1} \bar{2} \bar{1} \bar{1} \). The subtraction \( \cdots \bar{0} \bar{1} \bar{2} \bar{1} \bar{0} \bar{0} \bar{0} \). \( - \cdots \bar{0} \bar{1} \bar{2} \bar{1} \bar{0} \bar{1} \bar{2} \bar{1} \bar{1} \). \( = 1012 \), which is \( 27 + 3 + 2 \) in decimal notation, shows that \( 3^4x - x = 32 \). Thus \( x = \frac{80}{32} = 2/5 \).

c. The positive integers are precisely the \( p \)-adic integers that are eventually 0 to the left. Any subtraction sum of the form

\[
\begin{array}{cccc}
\cdots & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
- \cdots & 0 & 0 & a & b & c & \hline \\
\cdots & p-1 & p-1 & d & e & f \\
\end{array}
\]

finishes with recurring \( p-1 \) in the answer, because each 0 in the top line has to borrow 1 from the next place, causing 1 to be added in the column to the left in the second row, producing a sum \( 10 - 1 = p - 1 \) in \( p \)-adic notation. Conversely any subtraction sum

\[
\begin{array}{cccc}
\cdots & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
- \cdots & p-1 & p-1 & a & b & c & \hline \\
\cdots & 0 & 0 & d & e & f \\
\end{array}
\]

finishes with 0 to the left because 1 must always be borrowed, increasing \( p-1 \) to 10, giving an eventual computation \( 10 - 10 = 0 \) in each place.
d. The computation of the $p$-adic expansion of $a/b$ where $p \nmid b$ always gives a recurring string, by the pigeon hole principle, because at each stage in the division the $p$-adic remainder is one of the digits $\{1, \ldots, p-1\}$ and the calculation must repeat after some time. Equally, every $p$-adic integer $x$ with a recurring expansion of length $n$ is a rational integer because $p^n x - x = a$ is an integer, and now $x = \frac{a}{p^n - 1}$. The map $\mathbb{Z}(p) \to \mathbb{Z}_p$ specified by $\frac{a}{b} \mapsto (p$-adic expansion of $\frac{a}{b})$ is an injective ring homomorphism.

2. In this question consider the 10-adic topology on $\mathbb{Z}$, determined by the powers of the ideal (10), with completion the 10-adic integers $\mathbb{Z}^{\wedge}_{(10)}$, and also the 2-adic topology on $\mathbb{Z}$ with completion $\mathbb{Z}^{\wedge}_{(2)}$.

a. Show that a sequence of integers that is a Cauchy sequence in the 10-adic topology is also a Cauchy sequence in the 2-adic topology.

b. Show that the identity map $1 : \mathbb{Z} \to \mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}^{\wedge}_{(10)} \to \mathbb{Z}^{\wedge}_{(2)}$.

c. Determine whether the identity map $1 : \mathbb{Z} \to \mathbb{Z}$ extends to a ring homomorphism $\mathbb{Z}^{\wedge}_{(2)} \to \mathbb{Z}^{\wedge}_{(10)}$.

d. Using the fact that $\mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ as a product of rings, show that $\mathbb{Z}^{\wedge}_{(10)} \cong A \times B$ for certain rings $A, B$ that are also ideals of $\mathbb{Z}^{\wedge}_{(10)}$, with $A/(A \cap (10)) \cong \mathbb{Z}/2\mathbb{Z}$ and $B/(B \cap (10)) \cong \mathbb{Z}/5\mathbb{Z}$.

e. Show that $\mathbb{Z}^{\wedge}_{(10)}$ has just two maximal ideals, generated by 2 and 5.

f. Show that the composite morphism specified as the inclusion of the ideal $A \hookrightarrow \mathbb{Z}^{\wedge}_{(10)}$, followed by the ring homomorphism $\mathbb{Z}^{\wedge}_{(10)} \to \mathbb{Z}^{\wedge}_{(2)}$ of part b, is surjective. (Consider using Nakayama’s lemma.)

Solution: a. Taking the distance in the $m$-adic topology to be $d_m(a, b) = \frac{1}{m^t}$ if $m^t$ is the largest power of $m$ that divides $a - b$, if $(a_n)$ is a Cauchy sequence in the 10-adic topology then, given $\epsilon > 0$, we can find $u$ so that $\frac{1}{2^u} < \epsilon$. Now find $N$ so that $i, j \geq N$ implies $d_{10}(a_i, a_j) < \frac{1}{10^u}$, that is, $10^u|(a_i - a_j)$. Now $2^u|(a_i - a_j)$ so $d_2(a_i, a_j) \leq \frac{1}{2^u} < \epsilon$ for all $i, j \geq N$. This shows that $(a_n)$ is a Cauchy sequence in the 2-adic topology.

b. Regarding the completion as the set of equivalence classes of Cauchy sequences, the identity provides a map of sets

$$
\{10\text{-adic Cauchy sequences}\} \to \{2\text{-adic Cauchy sequences}\} \to \mathbb{Z}^{\wedge}_{(2)}
$$

by part a. Equivalent 10-adic Cauchy sequences are also 2-adic equivalent by a similar argument, so we get a map of sets $\mathbb{Z}^{\wedge}_{(10)} \to \mathbb{Z}^{\wedge}_{(2)}$, and it is a ring homomorphism because the identity map is.

c. The identity on $\mathbb{Z}$ does not extend to a ring homomorphism $f : \mathbb{Z}^{\wedge}_{(2)} \to \mathbb{Z}^{\wedge}_{(10)}$. Consider the composite of such an $f$ with the quotient map $\mathbb{Z}^{\wedge}_{(10)} \to \mathbb{Z}^{\wedge}_{(10)}/10\mathbb{Z}^{\wedge}_{(10)} \cong \mathbb{Z}/10\mathbb{Z}$ (the last isomorphism was done in class). Under this map 1 is sent to 1, which generates $\mathbb{Z}/10\mathbb{Z}$ as a ring, so the composite is surjective. The kernel contains 10, and $\mathbb{Z}^{\wedge}_{(2)}/10\mathbb{Z}^{\wedge}_{(2)} = \ldots$
\( \mathbb{Z}^\wedge_{(2)}/2\mathbb{Z}^\wedge_{(2)} \cong \mathbb{Z}/2\mathbb{Z} \) because 5 is invertible in \( \mathbb{Z}^\wedge_{(2)} \). This ring has size 2, so the composite cannot be surjective. Thus no such \( f \) can exist.

d. The decomposition of \( \mathbb{Z}/10\mathbb{Z} \) (assumed, but FYI it is a consequence of the Chinese Remainder Theorem) gives an expression 1 = \( e + (1-e) \) as a sum of two non-zero orthogonal idempotents, where \( e \) is the identity in \( \mathbb{Z}/2\mathbb{Z} \) and 1 - \( e \) is the identity in \( \mathbb{Z}/5\mathbb{Z} \). We did in class that \( \mathbb{Z}^\wedge_{(10)}/10\mathbb{Z}^\wedge_{(10)} \cong \mathbb{Z}/10\mathbb{Z} \), and we also did in class using Hensel’s lemma that there exists an idempotent \( f \in \mathbb{Z}^\wedge_{(10)} \) with \( f + 10\mathbb{Z}^\wedge_{(10)} = e \), giving a ring decomposition \( \mathbb{Z}^\wedge_{(10)} = A \times B \) where \( A = \mathbb{Z}^\wedge_{(10)} f \) and \( B = \mathbb{Z}^\wedge_{(10)} (1-f) \). The quotient map \( A \times B \to \mathbb{Z}/10\mathbb{Z} \) has kernel \( A \cap (10) \times B \cap (10) \) with the summands mapping to \( \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}/5\mathbb{Z} \), respectively, so \( A/(A \cap (10)) \cong \mathbb{Z}/2\mathbb{Z} \) and \( B/(B \cap (10)) \cong \mathbb{Z}/5\mathbb{Z} \).

e. Every element of \( \mathbb{Z}^\wedge_{(10)} \) not in \( (2) \) or \( (5) \) is invertible, by the same argument that showed that the completion at a maximal ideal is a local ring: if \( x \) is not in either ideal we can find \( y \) so that \( xy - 1 \in (10) \), so \( xy = 1 + a \) with \( a \in (10) \). Now \( (xy)^{-1} = 1 - a + a^2 - a^3 + \cdots \) and \( x^{-1} = y(xy)^{-1} \). From this it follows that if \( I \) is an ideal then \( I \subset (2) \cup (5) \). Now if \( I \) contains an element \( a \) not in \( (2) \) and \( b \) not in \( (5) \) then it contains \( a + b \) which lies in neither \( (2) \) nor \( (5) \), so is invertible, and \( I \) is the whole ring. This means that every ideal is contained in either \( (2) \) or \( (5) \) so these ideals are maximal and are the only such.

f. The composite \( \mathbb{Z}^\wedge_{(10)} \to \mathbb{Z}^\wedge_{(2)} \to \mathbb{Z}/2\mathbb{Z} \) is surjective because 1 is sent to 1, and this generates \( \mathbb{Z}/2\mathbb{Z} \). It gives rise to a surjective map of groups \( A/(A \cap (10)) \times B/(B \cap (10)) \to \mathbb{Z}/2\mathbb{Z} \), and the component \( B/(B \cap (10)) \cong \mathbb{Z}/5\mathbb{Z} \) it goes to 0. Thus the map \( A/(A \cap (10)) \to \mathbb{Z}/2\mathbb{Z} \) is surjective, as is \( A \to \mathbb{Z}/2\mathbb{Z} \). Now \( \mathbb{Z}^\wedge_{(2)} \) is a local ring, so that its Jacobson radical is \( 2\mathbb{Z}^\wedge_{(2)} \). Together with this radical, the image of \( A \) generates \( \mathbb{Z}^\wedge_{(2)} \). By Nakayama’s lemma, the image of \( A \) equals \( \mathbb{Z}^\wedge_{(2)} \), and the map is surjective.

3. Find how many cube roots each of the following numbers has in \( \mathbb{Z}^\wedge_{(7)} \): 1, 9, -4, 4, 12, 6. Also find how many cube roots each of the following numbers has in \( \mathbb{Z}^\wedge_{(5)} \): 1, 2, 3, 4, 5.

Solution. We find roots of \( f(x) = x^3 - t \) where \( t \) is prime to 7. Now \( f'(x) = 3x \) so if \( a \) in \( \mathbb{Z}^\wedge_{(7)} \) has \( a^3 \equiv t \) (prime to 7) then \( a \) is a unit (mod 7), as is \( f'(a) = 3a \). For each such \( a \), Hensel’s lemma applies and there is a cube root \( b \) of \( t \) with \( b \equiv a \) (mod 7). This means the number \( x \) cube roots of \( t \) in \( \mathbb{Z}^\wedge_{(7)} \) equals the number of cube roots of \( t \) in \( \mathbb{Z}/7\mathbb{Z} \). In \( \mathbb{Z}/7\mathbb{Z} \) the cubes of 1, 2, 3, 4, 5, 6 are 1, 1, 6, 1, 6, 6. This means the numbers 1, 6 both have 3 cube roots in \( \mathbb{Z}^\wedge_{(7)} \) and the other numbers 9, -4, 4, 12 have no cube root in \( \mathbb{Z}^\wedge_{(7)} \).

Doing the same thing module 5, the cubes of 1, 2, 3, 4 are 1, 3, 2, 4. This means that each \( 1, 2, 3, 4 \) has a unique cube root in \( \mathbb{Z}^\wedge_{(5)} \). The question probably should not have asked about cube roots of 5, but if \( x \in \mathbb{Z}^\wedge_{(5)} \) lies in \( (5)^d \) then \( x^3 \) lies in \( (5)^{3d} \). From this we see that \( x^3 = 5 \) has no solutions, because \( 5 \notin (5)^{3d} \) with \( d \geq 1 \).

4. Let \( I \) be an ideal of \( R \). Consider the polynomial \( f(x) = 3x^4 + x^2 + 5 \) as a function \( R \to R \). Show that \( f \) is continuous in the \( I \)-adic topology on \( R \). (The \( I \)-adic topology on \( R \) is given by the distance function determined by the powers of \( I \).)
Solution. We use the distance function \( d(u,v) = \frac{1}{2^n} \) if \( u-v \in I^n - I^{n+1} \), and write \( |u| = d(u,0) \). We show first that the function \( x^r \) is continuous. Given \( \epsilon > 0 \) take \( \delta = \epsilon \). Now if \( |u| < \delta \) then \( u \in I^N \) where \( \frac{1}{2^n} < \delta \), and \( d(x^r,(x+u)^r) = |(x+u)^r - x^r| = |uv| < \epsilon \) (for some \( v \)) because \( uv \in I^N \) also \((I^N) \) is an ideal. This shows that \( x^r \) is continuous.

We next show that if \( f \) and \( g \) are continuous functions then \( f + g \) is continuous. For each \( x \), given \( \epsilon > 0 \) we can find \( \delta \) so that \( |u| < \delta \) implies both \( |f(x+u) - f(x)| < \epsilon \) and \( |g(x+u) - g(x)| < \epsilon \). This means that \( f(x+u) - f(x) \in I^N \) and \( g(x+u) - g(x) \in I^N \) for some \( N \) with \( \frac{1}{2^n} < \epsilon \) and now \( f(x+u) + g(x+u) - (f(x) + g(x)) = (f(x+u) - f(x)) + (g(x+u) - g(x)) \in I^N \) so \( |f(x+u) + g(x+u) - (f(x) + g(x))| < \epsilon \). This shows that \( f + g \) is continuous. Scalar multiplication is continuous, similarly. Putting this together we see that polynomials are continuous.

5. For a category \( C \) and commutative ring \( R \) we may take the \( R \)-linear category \( RC \) to have the same objects as \( C \), and with \( \text{Hom}_{RC}(x,y) = R \text{Hom}_C(x,y) \), the set of formal linear combinations of morphism \( x \to y \) in \( C \). Composition is \( R \)-bilinear. The constant functor \( R : RC \to R \text{-mod} \) is the functor that assigns \( R \) to each object of \( C \), and the identity map \( 1_R \) to each morphism of \( C \).

a. Let \( C \) be the category \( \bullet \leftarrow \bullet \to \bullet \) with three objects, and five morphisms that are the two morphisms shown and the three identity morphisms for the objects. Show that the constant functor on \( C \) is representable as a linear functor \( RC \to R \text{-mod} \).

b. Let \( D \) be the category \( \bullet \to \bullet \leftarrow \bullet \) with three objects, and five morphisms, with the two non-identity morphisms pointing in the opposite direction to the last example. Show that the constant functor is not representable.

c. Show that the inverse limit functor \( \varprojlim : \text{Fun}(D,R \text{-mod}) \to R \text{-mod} \) is representable, represented by the constant functor.

Solution. a. Label the three objects \( a,b,c \) from left to right, and the non-identity morphisms \( \alpha : b \to a \) and \( \beta : b \to c \). We claim that the constant functor is represented by object \( b \). This is because \( \text{Hom}_{RC}(b,x) \cong R \) for each object \( x \), and each morphism of \( C \) is sent by this functor to an isomorphism. Specifically, \( \text{Hom}_{RC}(b,a) = R\alpha \), \( \text{Hom}_{RC}(b,b) = R1_b \) and \( \text{Hom}_{RC}(b,c) = R\beta \). The functorial effect on \( \alpha \) is postcomposition with \( \alpha \), namely \( \alpha_* : \text{Hom}_{RC}(b,b) \to \text{Hom}_{RC}(b,a) \), so \( \alpha_* (1_b) = \alpha \), and it is similar with \( \beta \). This functor is thus naturally isomorphic to the constant functor, by a natural isomorphism that sends each of \( \alpha, 1_b, \beta \) to \( 1 \) in \( R \).

b. Label the three objects \( a,b,c \) from left to right. If the constant functor were representable, it would be representable by one of \( a,b,c \). The representable functor at \( a \) is non-zero only on \( a \) and \( b \), the representable functor at \( b \) is non-zero only on \( b \), and the representable functor at \( c \) is non-zero only on \( b \) and \( c \). None of these is the constant functor, so it is not representable.
c. We have seen in class exactly that $\varprojlim F \cong \text{Hom}_{\text{Fun}}(R, F)$ where $R$ is the constant functor on $\mathcal{D}$, Fun is short for $\text{Fun}(\mathcal{D}, R\text{-mod})$ and the Hom denotes natural transformations. Thus $\varprojlim$ and $\text{Hom}_{\text{Fun}}(R, -)$ are naturally isomorphic functors, and are representable.

6. Let $\text{Fun}(\mathcal{C}, \text{abgps})$ be the category of functors from $\mathcal{C}$ to abelian groups, with natural transformations as morphisms. We may take as a definition that a sequence $F_1 \to F_2 \to F_3$ in $\text{Fun}(\mathcal{C}, \text{abgps})$ is exact if and only if, for all objects $X$ in $\mathcal{C}$, the sequence of abelian groups $F_1(X) \to F_2(X) \to F_3(X)$ is exact. This is equivalent to other possible definitions of exactness. We may regard the inverse limit construction as a functor $\varprojlim : \text{Fun}(\mathcal{C}, \text{abgps}) \to \text{abgps}$.

a. Let $\mathcal{C}$ be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ with three objects, and five morphisms that are the two morphisms shown and the three identity morphisms for the objects. Show that the functor $\varprojlim : \text{Fun}(\mathcal{C}, \text{abgps}) \to \text{abgps}$ is exact.

b. Let $\mathcal{D}$ be the category $\bullet \rightarrow \bullet \leftarrow \bullet$ with three objects, and five morphisms, with the two non-identity morphisms pointing in the opposite direction to the last example. Show (by example, or by giving a reason) that the functor $\varprojlim : \text{Fun}(\mathcal{D}, \text{abgps}) \to \text{abgps}$ is not exact in general.

Solution. a. The constant functor $R$ is representable and hence projective in $\text{Fun}(\mathcal{C}, \text{abgps})$, by something we did in class. This means that $\text{Hom}_{\text{Fun}}(R, -) \cong \varprojlim$ is exact.

b. We have seen that $R$ is not representable in this case, and in fact it is not projective. We could see this from our knowledge of representations of the quiver $\bullet \to \bullet \leftarrow \bullet$. A more rudimentary approach is to produce a short exact sequence of functors $0 \to F_1 \to F_2 \to F_3 \to 0$ on which $\varprojlim$ is not exact. Let $F_1$ be the functor described by $0 \to R \leftarrow 0$, meaning that $F_1(a) = 0$, $F_1(b) = R$ and $F_1(c) = 0$. Similarly, let $F_2$ be $(R \to R \leftarrow 0) \oplus (0 \to R \leftarrow R)$ and let $F_3$ be $R = R \to R \leftarrow R$. All morphisms in describing these functors and the short exact sequence are either $1_R$ or $0$. Now $\varprojlim F_3 = R$ and $\varprojlim F_2 = 0$, so $\varprojlim F_2 \to \varprojlim F_3$ is not surjective. This shows that $\varprojlim$ is not exact.