**Math 8212 Commutative and Homological Algebra 2 Spring 2022**

**Homework Assignment 4** Due **Saturday 5/7/2022**, uploaded to Gradescope.
Each question part is worth 1 point. There are 8 question parts. You are on target for an A if you make a genuine attempt on at least half of them. This homework has fewer parts than previous homeworks. If you can find a way to do the calculation of your overall score so that it comes to be more than 50% (e.g. by weighting each of the four homeworks so that they count equally, or by something else), I will accept that.

1. Although we did not define explicitly the higher differentials in a spectral sequence, it is possible to deduce what they must be in this example. Consider the double complex

$$\begin{array}{ccccc} \mathbb{Z} & \xleftarrow{p} & \mathbb{Z} & \longleftarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{p} & \mathbb{Z} \end{array}$$

with terms that are 0 except as shown, the nonzero terms being in bidegrees

$$(0,0),(1,0),(-1,1) \text{ and } (0,1).$$

a. Compute the homology of the total complex of this double complex. [You may want to use standard facts having to do with structure of finitely generated modules over a Euclidean domain, Smith Normal Form etc, including the fact that the order of a quotient of a free abelian group by the subgroup spanned by columns of a square matrix is the determinant of that matrix.]

b. Filtering the double complex by rows (so that the modules in each row describe the quotient of two consecutive terms in the filtration), we get a spectral sequence. Find all the numbers $n$ for which $E^n = E^{n+1}$. Determine whether the naturally given grading on the $E^\infty$ term is the same as the naturally given grading on the homology of the total complex from a.

c. Filtering the double complex by columns (so that the modules in each column describe the quotient of two consecutive terms in the filtration), we get a spectral sequence. Find all the numbers $n$ for which $E^n = E^{n+1}$. Determine whether the naturally given grading on the $E^\infty$ term is the same as the naturally given grading on the homology of the total complex from a.

d. Describe how to construct a spectral sequence for which $E^5 \neq E^6 = E^\infty$.

Solution. a. The total complex is a complex $0 \leftarrow \mathbb{Z}^2 \xleftarrow{\alpha} \mathbb{Z}^2 \leftarrow 0$ where the matrix of $\alpha$ is $\begin{pmatrix} p & 0 \\ 1 & p \end{pmatrix}$. We see that $\alpha$ is injective (non-zero determinant) so the top homology is 0. The bottom homology is the quotient of $\mathbb{Z}^2$ by the span of the columns of the matrix, which is equivalent by row and column operations to

$$\begin{pmatrix} p & 0 \\ 1 & p \end{pmatrix} \sim \begin{pmatrix} 0 & -p^2 \\ 1 & p \end{pmatrix} \sim \begin{pmatrix} -p^2 & 0 \\ p & 1 \end{pmatrix} \sim \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

A result in Math8202 says these operations do not change the factor groups so the bottom homology is $\mathbb{Z}/p^2\mathbb{Z}$.
b. The $E^0$ and $E^1$ are

$$\begin{array}{ccccc} \mathbb{Z} & \xleftarrow{p} & \mathbb{Z} & \longleftarrow & 0 \\ \\ 0 & \longleftarrow & \mathbb{Z} & \xleftarrow{p} & \mathbb{Z} \end{array}, \qquad \begin{array}{ccc} \mathbb{Z}/p\mathbb{Z} & 0 & 0 \\ \big\downarrow & \big\downarrow & \big\downarrow \\ 0 & \mathbb{Z}/p\mathbb{Z} & 0 \end{array}$$

and differentials are 0 from that page onwards because they map between positions where at least one term is 0. Thus $E^1 = E^2 = \cdots = E^\infty$ so the $n$ with $E^n = E^{n+1}$ are the $n \geq 1$.
c. The $E^0$ and $E^2$ are

$$\begin{array}{ccc} \mathbb{Z} & \mathbb{Z} & 0 \\ \big\downarrow & \big\downarrow{\scriptstyle 1} & \big\downarrow \\ 0 & \mathbb{Z} & \mathbb{Z} \end{array}, \qquad \begin{array}{ccccc} \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \\ \\ 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z} \end{array}, \qquad \begin{array}{ccc} \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} \end{array}$$

where $d_2$ maps the bottom right $\mathbb{Z}$ to the top left $\mathbb{Z}$. Subsequent differentials are 0 because they go between positions where at least one term is 0, so we deduce $E^1 = E^2 \neq E^3 = E^4 = \cdots = E^\infty$ and the non-zero $d_2$ component must be $p^2$ to give the correct answer for $E^\infty$. The $n$ asked for are $n = 1$ and $n \geq 3$.
d. A similar double complex with 5 staggered rows will give $E^5 \neq E^6 = E^\infty$.

2. (Similar to Exercise 10.12 of Eisenbud) Let $S = k[x_1, \ldots, x_r]$ be a polynomial ring over a field $k$ in $r$ variables, where the indeterminate $x_i$ has degree $d_i$. Let $M$ be a finitely generated graded $S$-module and put $H_M(n) = \dim_k M_n$ and $h_M(t) = \sum_{n \geq 0} H_M(n) t^n$.

a. Show that $h_M(t)$ is a rational function of $t$, and that in fact $h_M(t)$ may be written as a polynomial divided by $\prod_{i=1}(1 - t^{d_i})$.

b. Show that there is a number $d$ (which may be taken to be the least common multiple of the degrees $d_i$) such that for each $s$, $H_M(dn + s)$ agrees with a polynomial in $n$ for all $n >> 0$; that is, $H_M(n)$ is an 'almost PORC function' of $n$.

Solution a. This part was done in class. Consider the exact sequence

$$0 \to K \to M \xrightarrow{x_r} M \to L \to 0.$$

Then in each degree $n$ we have $\dim K_n - \dim M_n + \dim M_{n+d_r} - \dim L_{n+d_r} = 0$. Multiply by $t^{n+d_r}$ and sum to get $t^{d_r} h_K(t) - t^{d_r} h_M(t) + h_M(t) - h_L(t) = g(t)$ for some polynomial $g(t)$. This gives

$$(1 - t^{d_r}) h_M(t) = h_L(t) - t^{d_r} h_K(t) = \frac{u(t)}{\prod_{i=1}^{r-1}(1 - t^{d_i})}$$

by induction on $r$, because $K$ and $L$ are annihilated by $x_r$ and so are $k[x_1, \ldots, x_{r-1}]$-modules, where $u$ is some further polynomial. We obtain

$$h_M(t) = \frac{u(t)}{\prod_{i=1}^{r}(1 - t^{d_i})}$$

b. Let $d$ be the l.c.m. of $\{d_1, \ldots, d_r\}$, We can write $\frac{1}{(1-t^{d_i})} = \frac{q_i(t)}{(1-t^d)}$ where $q_i(t) = 1 + t^{d_i} + \cdots + t^{(d-d_i)}$ and this allows us to write $h_M(t)$ as a polynomial divided by $(1-t^d)^r$. The function $\frac{1}{(1-t^d)^r}$ is almost PORC with modulus $d$ where the coefficient of $t^{dn+s}$ is 0 unless $s = 0$, in which case it equals the coefficient of $t^n$ in $\frac{1}{(1-t)^r}$, which is polynomial for $n >> 0$. Multiplying by a polynomial does not change the almost PORC property.

3. Let $k$ be a field. The ring $A = k[x, y]/(x^2 - y^3)$ that we studied in class has maximal ideal $\mathfrak{m} = (\bar{x}, \bar{y})$, where bars denote the images of elements in $A$. We saw in class that the ideal $(\bar{y})$ is $\mathfrak{m}$-primary. For any $\mathfrak{m}$-primary ideal $J$ we may form the (Hilbert-)Samuel function $\chi_J(n) = \dim_k(A/J^n)$. Note that if we were to localize at $\mathfrak{m}$ we would have $A_{\mathfrak{m}}/(J_{\mathfrak{m}})^n \cong A/J^n$, so we don't have to localize before computing the dimension.

a. Compute the values of $\chi_{\mathfrak{m}}(n)$. Find the polynomial $f(t)$ such that $\chi_{\mathfrak{m}}(n) = f(n)$ for large $n$.

b. Compute the values of $\chi_{(\bar{y})}(n)$. Find the polynomial $f(t)$ such that $\chi_{(\bar{y})}(n) = f(n)$ for large $n$.

Solution a. $A$ has $k$-basis $\{1, \bar{x}, \bar{y}, \bar{x}\bar{y}, \bar{y}^2, \bar{x}\bar{y}^2, \ldots\}$. The power $\mathfrak{m}^n$ has basis this list from $\bar{x}\bar{y}^{n-1}$ onwards, so the values of $\dim A/\mathfrak{m}^n$ when $n = 0, 1, 2, \ldots$ are $0, 1, 3, 5, 7, \ldots$ so $f(t) = 2t - 1$.
b. In this case $(\bar{y})^n$ is spanned by the elements in the list from $\bar{y}^n$ onwards, so the values of $\dim A/(\bar{y})^n$ when $n = 0, 1, 2, \ldots$ are $0, 2, 4, 6, \ldots$ so $f(t) = 2t$.