Chapter 4: Integrality and some other things

Given a commutative ring R, we study R-algebras A so that A is finitely generated (or finite) as an R-module.

Examples:

1. The group ring RG. = free R-module with the elements of G as a basis. This is finite over R if G is finite.

2. Let R = Z contained in the rational numbers Q. What subrings of Q are finitely generated as Z-modules?

e.g. is $\mathbb{Z}[\frac{1}{2}]$ finitely generated? No

The only such subring is $\mathbb{Z}$.

3. Inside Q[i], consider Z[i], Z[2i], Z[i/2].

$\mathbb{Z}[i]$ is finitely generated over $\mathbb{Z}$.

$\mathbb{Z}[2i] \subseteq \mathbb{Z}[i]$ is also finitely generated over $\mathbb{Z}$.

$\mathbb{Z}[\frac{i}{2}]$ is not fin. gen over $\mathbb{Z}$.

Defn. A commutative R-algebra A is finite over R if it is finitely generated as an R-module.

Does your neighbor know what an R-algebra is?

Yes ✓

No

Definition R is a commutative ring. An R-algebra A is a ring with a ring homomorphism $\phi: R \to Z(A)$ (so $1_R \mapsto 1_A$).

A is an R-module via $r.a := \phi(r)a$

We learn:

- What condition on elements of A produce this finiteness condition
- What are the properties of integers.

Definitions. Let S be an R-algebra.

S should be commutative.

An element s of S is integral over R if and only if $p(s) = 0$ in S where $p(x) \in R[x]$ is a monic polynomial.

monic: the leading coefficient is 1

The algebra S is integral over R if.

every element of S is integral over R.

Not clear: $\mathbb{Z}\left[\frac{1+i\sqrt{3}}{2}\right] \subseteq \mathbb{C}$ is integral over $\mathbb{Z}$.

Examples: $1 + i\sqrt{3}$ and $(1+i\sqrt{3})/2$.

$$(1+i\sqrt{3})^2 = 1 - 3 + 2i\sqrt{3}$$
$$= -2 + 2i\sqrt{3}$$
$$= 2(1+i\sqrt{3}) - 4$$

$1+i\sqrt{3}$ is a root of of $X^2 - 2X + 4$. so is integral over $\mathbb{Z}$.

$$\left(\frac{1+i\sqrt{3}}{2}\right)^2 = \frac{1+i\sqrt{3}}{2} - 1$$

$\frac{1+i\sqrt{3}}{2}$ is a root of $X^2 - X + 1$ so is also integral over $\mathbb{Z}$.

$1+i\sqrt{5}$ is integral over $\mathbb{Z}$

$\frac{1+i\sqrt{5}}{2}$ is not integral over $\mathbb{Z}$.

Goal: the integral elements form a sub ring. of S.

Corollary 4.6 plus. Let S be an R-algebra.
TFAE for s in S
(a) s is integral over R.
(b) $R[s]$ is contained in an R-submodule M of S, finitely generated over R, with $sM \subseteq M$
(c) there exists an S-module N and a finitely generated R-submodule M of N, not annihilated by nonzero element of S, such that $sM \subseteq M$.

Proof (a) $\Rightarrow$ (b). Take

$$M = R[s] \quad (= \phi(R)[s])$$
$$\subseteq S.$$

Then $s \cdot R[s] \subseteq R[s]$.

Show $R[s]$ is finitely generated over R. It is generated by

$$1, s, s^2, s^3, \ldots$$

If $p(s) = 0$

$$p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

then $s^n = -(b_{n-1}s^{n-1} + \cdots + b_0)$

so s^n is not needed as a generator of $R[s]$.

$R[s]$ is generated by

$$1, s, s^2, \ldots, s^{n-1}. \qquad \square$$

Goal: the integral elements form a sub ring.

Corollary 4.6 plus. Let S be an R-algebra. TFAE for s in S

(a) s is integral over R.

(b) $R[s]$ is contained in an R-submodule M of S, finitely generated over R, with $sM \subseteq M$

(c) there exists an S-module N and a finitely generated R-submodule M of N, not annihilated by nonzero element of S, such that $sM \subseteq M$.

Corollary = Theorem 4.2.
Elements of S integral over R form an R-subalgebra.

The extra things in Eisenbud's book