Category Theory

Eisenbud's Appendix 5 has the right topics but is brief with a shortage of examples.

Definition. A category C is the specification of

1. A class of things called 'objects'
   $x \in Ob(\mathcal{C})$ means x is an object of $\mathcal{C}$.
2. For each pair $x, y \in Ob(\mathcal{C})$ we have a set $Hom_{\mathcal{C}}(x,y)$ of things called morphisms.
3. A rule of composition

$$Hom(y,z) \times Hom(x,y) \longrightarrow Hom(x,z)$$
$$(g\ ,\ f) \longmapsto gf \ (\text{or } g \circ f)$$

so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in Ob(\mathcal{C})$, there exists a morphism $1_x : x \to x$ so that $f 1_x = f \ \forall f : x \to y$, $1_x j = j \ \forall j : w \to x$

Morphism notation If $f \in Hom_{\mathcal{C}}(x,y)$ we write $f : x \to y$ to denote this. x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples

1. Set = category with objects = sets, morphisms = maps of sets.

Top = category of topological spaces, morphisms = continuous maps

Group: morphisms = group homoms

R-mod: Objects are R-modules, Morphisms are R-module homomorphisms.

2. A poset P may be regarded as a category P with Ob(P) = elements of P, $\exists$ unique morphism $x \to y$ $\Rightarrow x \leq y$ in P.

Pre-class Warm-up!!!

Suppose $f : M \to N$ is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms $g, h : L \to M$, if $fg = fh$ then $g = h$.

B. For all pairs of homomorphisms $g, h : N \to Q$, if $gf = hf$ then $g = h$.

C. Neither of the above.

$B \iff f$ is onto.

A. $L \underset{h}{\overset{g}{\rightrightarrows}} M \xrightarrow{f} N$ $\quad fg = fh \Rightarrow g = h$

Proposition $A \iff f$ is 1-1.

Proof "$A \Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero. Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A" 1-1 $\iff \ker f = 0$. If $fg = fh$ then $\forall x \in L$, $f g(x) = f h(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\Leftrightarrow \exists$ a morphism $g: y \to x$ so that $gf = 1_x$ and $fg = 1_y$.

The following is not equivalent to f being an isomorphism.

or $\Leftarrow$ f is 1-1 and onto ?

$\Updownarrow$

A, say 'f is a monomorphism'

I just suggested it for the purposes of discussion.

B say 'f is an epimorphism'

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may construct a category $\mathcal{G}$ with only one object $*$ and where $Hom(*,*) = G$

composition := multiplication in G!

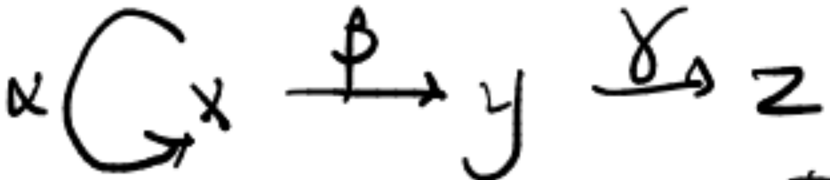
If M is a monoid we construct a category $\mathcal{M}$ with one object $*$

$Hom(*,*) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)



Construct a category F(Q) where

Ob(F(Q)) = vertices of Q

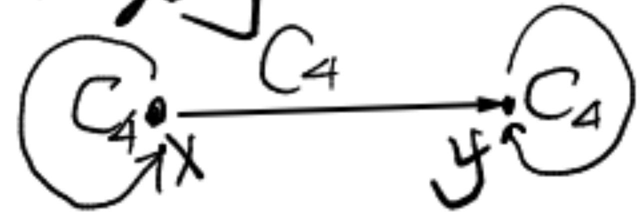
Morphisms = all possible words in the edges of Q where the end of a symbol = start of next.

Example Ob = $\{x, y, z\}$

Morphisms = $\{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \dots\}$

FI = the category with Objects = Finite sets,

Schematic description of a category with $Ob(\mathcal{C}) = \{x, y\}$



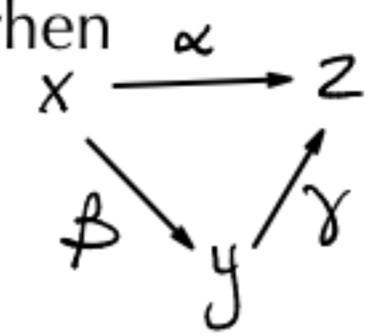
4 morphisms $x \to y$, $End(x) = C_4$, $End(y) = C_4$

$End(x) = Hom_{\mathcal{C}}(x, x)$

Question:

How many morphisms does F(Q) have when

Q =



A 3

B 4

C 5

D 6

E 7

F 8

G Infinitely many.

$$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$$

$$\{1, a, a^2, a^3\}$$

Composition

$$a^2 \circ a_x = a^3$$

$$a_x^2 \circ a_x = a_x^3$$

Constructions.

Question: Let I be the poset $0 \to 1$

How many morphisms does I x I have?

A 1
B 2
C 4
D 6
E 8
F 9

Functors

Definition.

Question: did we need to put in $T(1) = 1$ always, or did it follow from the other

Examples

Inclusion

Forgetful functor

If G and H are groups

If P and Q are posets

If X is a set let $R(X) =$

If M is a right R-module, L is a left R-module, we have functors

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

If G is a group (or a monoid) a functor F : G -> R-mod

Definition. A category C is small if $Ob(C)$ is a set.

Example:
SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

The functors between the category of representations of a group G and the category of RG-modules.

If Q is a directed graph (= a quiver), a representation of Q over R is

It is the same thing as a functor F(Q) -> R-mod

A homomorphism of quiver representations

Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F , G : C -> D by functors.
A natural transformation