Category Theory

Eisenbud's Appendix 5 has the right topics but is brief with a shortage of examples.

Definition. A category C is the specification of

1. A class of things called 'objects'
   $x \in Ob(\mathcal{C})$ means x is an object of $\mathcal{C}$.
2. For each pair $x, y \in Ob(\mathcal{C})$ we have a set $Hom_{\mathcal{C}}(x,y)$ of things called morphisms.
3. A rule of composition
   $$Hom(y,z) \times Hom(x,y) \longrightarrow Hom(x,z)$$
   $$(g\ ,\ f) \longmapsto gf \ (\text{or } g \circ f)$$

so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in Ob(\mathcal{C})$, there exists a morphism $1_x : x \to x$ so that $f 1_x = f \ \forall f : x \to y$, $1_x j = j \ \forall j : w \to x$

Morphism notation If $f \in Hom(x,y)$ we write $f : x \to y$ to denote this. x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples

1. Set = category with objects = sets, morphisms = maps of sets.

   Top = category of topological spaces, morphisms = continuous maps

   Group: morphisms = group homoms

   R-mod: Objects are R-modules, Morphisms are R-module homomorphisms.

2. A poset P may be regarded as a category P with Ob(P) = elements of P, $\exists$ unique morphism $x \to y$ $\Rightarrow x \leq y$ in P.

Pre-class Warm-up!!!

Suppose f : M -> N is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms g, h : L -> M, if fg = fh then g = h.

B. For all pairs of homomorphisms g, h : N -> Q, if gf = hf then g = h.

C. Neither of the above.

B $\Leftrightarrow$ f is onto.

A. $L \underset{h}{\overset{g}{\rightrightarrows}} M \xrightarrow{f} N$ $\quad fg = fh \Rightarrow g = h$

Proposition A $\Leftrightarrow$ f is 1-1.

Proof "A $\Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero. Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A" 1-1 $\Leftrightarrow$ $\ker f = 0$

If $fg = fh$ then $\forall\, x \in L$, $f g(x) = f h(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\iff \exists$ a morphism $g: y \to x$ so that $gf = 1_x$ and $fg = 1_y$.

The following is not equivalent to f being an isomorphism.

or $\iff$ f is 1-1 and onto ?

$\Updownarrow$

A, say 'f is a monomorphism'

I just suggested it for the purposes of discussion.

B say 'f is an epimorphism'

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may construct a category $\mathcal{G}$ with only one object $*$ and where $\mathrm{Hom}(*,*) = G$

composition := multiplication in G!

If M is a monoid we construct a category $\mathcal{M}$ with one object $*$

$\mathrm{Hom}(*,*) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)

$$\begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & y \arrow[r, "\gamma"] & z \end{tikzcd}$$

Construct a category $F(Q)$ where

$Ob(F(Q))$ = vertices of Q

Morphisms = all possible words in the edges of Q where the end of a symbol = start of next.

Example $Ob = \{x, y, z\}$

Morphisms $= \{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \dots\}$

FI = the category with Objects = Finite sets,

Schematic description of a category with $Ob(\mathcal{C}) = \{x, y\}$

$$\begin{tikzcd} x \arrow[loop left, "C_4"] \arrow[r, "C_4"] & y \arrow[loop right, "C_4"] \end{tikzcd}$$

4 morphisms $x \to y$, $End_{\mathcal{C}}(x) = C_4$, $End(y) = C_4$

$End_{\mathcal{C}}(x) = Hom_{\mathcal{C}}(x, x)$

Question:

How many morphisms does $F(Q)$ have when

$$Q = \begin{tikzcd} x \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & z \\ & y \arrow[ur, "\gamma"'] & \end{tikzcd}$$

A 3

B 4

C 5

D 6

E 7

F 8

G Infinitely many.

$$\begin{tikzcd} \bullet \arrow[loop left, "C_4"] \arrow[r, "C_4"] & \bullet \arrow[loop right, "C_4"] \end{tikzcd}$$

$\{1_y, a_y \ \ldots\}$

$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$

$\{1, a, a^2, a^3\}$

Composition

$$a^2 \circ a_x = a^3$$

$$a_x^2 \circ a_x = a_x^3$$

Pre-class Warm-up!

How many morphisms are there in the free category generated by the quiver

$$\begin{tikzcd} w \arrow[r, "\alpha"] & x \arrow[r, "\beta"] & y & z \arrow[l, "\gamma"'] \end{tikzcd}$$

A 3

B 4

C 7

D 8

Constructions.

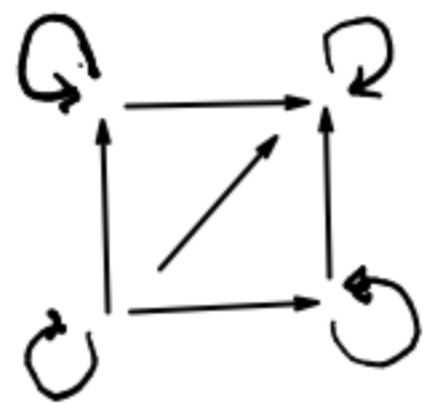
- The product of two categories $\mathcal{C}$, $\mathcal{D}$ is the category $\mathcal{C}\times\mathcal{D}$ with objects (c,d), $c\in Ob(\mathcal{C})$, $d\in Ob(\mathcal{D})$, morphisms $(c,d)\xrightarrow{(f,g)}(c',d')$ where $f: c\to c'$ in $\mathcal{C}$, $g: d\to d'$ in $\mathcal{D}$.

- If $\mathcal{C}$ is a category, the opposite category is $\mathcal{C}^{op}$ with $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and morphisms $\bar{\alpha}$ where α is a morphism in $\mathcal{C}$. If $\alpha: x\to y$ then $\bar{\alpha}: y\to x$.
  $\bar{\beta}: z\to y$, $\beta: y\to z$
  $\bar{\alpha}\bar{\beta} := \overline{\beta\alpha}$

Question: Let I be the poset $0\xrightarrow{\alpha}1$

How many morphisms does $I\times I$ have?

A 1
B 2
C 4
D 6
E 8
F 9 ✓



Functors

Definition. $\mathcal{C}, \mathcal{D}$ are categories.

the specification of

- $\forall\ x \in Ob(\mathcal{C})$, an object $T(x) \in Ob(\mathcal{D})$
- $\forall$ morphisms $f : x \to y$ in $\mathcal{C}$, a morphism $T(f) : T(x) \to T(y)$ in $\mathcal{D}$

so that

1. $T(fg) = T(f)\,T(g)$
2. $T(1_x) = 1_{T(x)} \quad \forall x \in Ob(\mathcal{C})$

Question: did we need to put in $T(1) = 1$ always, or did it follow from the other axioms

Examples $F : \text{AbGroups} \to \text{Groups}$

Inclusion $F(A) = A$, $F(f) = f$

Forgetful functor like $F : \text{Groups} \to \text{Set}$

$F(G) = G$ regarded as a set

or $k[x]$-mod $\to$ k-mod = vector spaces over k.

If G and H are groups we get categories $\mathcal{G}, \mathcal{H}$. A functor $F : \mathcal{G} \to \mathcal{H}$ is 'the same thing as' a group homomorphism $G \to H$.

If P and Q are posets a functor $F : P \to Q$ is 'the same thing as' an order preserving map $P \to Q$.

If X is a set let $R(X)$ = free R-module with X as a basis.

R is a commutative ring

$R(-)$ is a functor $\text{Set} \to R\text{-mod}$

$F(-) : \text{Set} \to \text{Group}$ is a functor. ($F(X)$ = free group generated by X)

If M is a right R-module, L is a left R-module, we have functors

$M \otimes - \;:\; R\text{-mod} \to \text{Ab Group}$

$\mathrm{Hom}_R(L, -) : R\text{-mod} \to \text{Ab Group}$

R-mod = category of left R-modules
mod-R = right

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$
$\mathcal{C} \to \mathcal{D}$

A contravariant functor has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$.

If G is a group (or a monoid) a functor
F : G -> R-mod

A repn of G over R is a homomorphism $G \to GL(V)$ for some R-module V.

$F : G \to R\text{-mod}$ is
$F(*) = V$
$\forall\ g : * \to *$, $F(g) : V \to V$ is an R-module homomorphism.

$F(g g^{-1}) = F(1_*) = 1_V : V \to V$
$= F(g)\, F(g^{-1})$

$F(g^{-1}) = F(g)^{-1}$

Definition. A category C is small if Ob(C) is a set.

Example:
SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$\rho : G \to GL(V)$ $\sigma : G \to GL(W)$

$\rho \to \sigma$

Morphisms in the category Reps of G are R-linear maps $\theta : V \to W$ so that $\theta(\rho(g)(v)) = \sigma(g)(\theta(v))$.

i.e the square

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\theta"'] & V \arrow[d, "\theta"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

commutes $\forall g \in G$.

The functors between the category of representations of a group G and the category of RG-modules.

Reps of G are the same thing as RG-modules

RG = group ring of G
= free R-module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

$$\text{Reps of } G \rightleftarrows RG\text{-mod}$$

Given $\rho : G \to GL(V)$ get an RG-module V

$$\left(\sum_{g \in G} a_g g\right) \cdot v := \sum_{g \in G} a_g \, \rho(g)(v)$$

If Q is a directed graph (= a quiver), a representation of Q over R is

It is the same thing as a functor $F(Q) \to R\text{-mod}$

A homomorphism of quiver representations

Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F , G : C -> D by functors.
A natural transformation