Category Theory
Eisenbud's Appendix 5 has the right topics but is brief with a shortage of examples.

Definition. A category C is the specification of

1. A class of things called 'objects' $x \in Ob(\mathcal{C})$ means x is an object of $\mathcal{C}$.
2. For each pair $x, y \in Ob(\mathcal{C})$ we have a set $Hom_{\mathcal{C}}(x,y)$ of things called morphisms.
3. A rule of composition

$$Hom(y,z) \times Hom(x,y) \longrightarrow Hom(x,z)$$
$$(g\ ,\ f) \longmapsto gf \ (\text{or } g \circ f)$$

so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in Ob(\mathcal{C})$, there exists a morphism $1_x : x \to x$ so that $f 1_x = f \ \forall f : x \to y$, $1_x j = j \ \forall j : w \to x$

Morphism notation If $f \in Hom(x,y)$ we write $f : x \to y$ to denote this. x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples

1. **Set** = category with objects = sets, morphisms = maps of sets.

Top = category of topological spaces, morphisms = continuous maps

Group: morphisms = group homoms

R-mod: Objects are R-modules. Morphisms are R-module homomorphisms.

2. A poset P may be regarded as a category $\mathcal{P}$ with $Ob(\mathcal{P})$ = elements of P, $\exists$ unique morphism $x \to y$ $\Leftrightarrow x \leq y$ in P.

Pre-class Warm-up!!!

Suppose f : M -> N is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms g, h : L -> M, if fg = fh then g = h.

B. For all pairs of homomorphisms g, h : N -> Q, if gf = hf then g = h.

C. Neither of the above.

$B \Leftrightarrow f$ is onto.

A. $L \underset{h}{\overset{g}{\rightrightarrows}} M \xrightarrow{f} N$ $\quad fg = fh \Rightarrow g = h$

Proposition $A \Leftrightarrow f$ is 1-1.

Proof "$A \Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero. Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A" 1-1 $\Leftrightarrow \ker f = 0$

If $fg = fh$ then $\forall x \in L$, $fg(x) = fh(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\Leftrightarrow \exists$ a morphism $g: y \to x$ so that $gf = 1_x$ and $fg = 1_y$.

The following is not equivalent to f being an isomorphism.

or $\Leftarrow$) f is 1-1 and onto ?

$\Updownarrow$

A, say 'f is a monomorphism'

I just suggested it for the purposes of discussion.

B say 'f is an epimorphism'

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may construct a category $\mathcal{G}$ with only one object $*$ and where $\mathrm{Hom}(*,*) = G$

composition := multiplication in G!

If M is a monoid we construct a category $\mathcal{M}$ with one object $*$

$\mathrm{Hom}(*,*) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)

$$\begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & y \arrow[r, "\gamma"] & z \end{tikzcd}$$

Construct a category $F(Q)$ where

$Ob(F(Q))$ = vertices of Q

Morphisms = all possible words in the edges of Q where the end of a symbol = start of next.

Example $Ob = \{x, y, z\}$

Morphisms $= \{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \dots\}$

FI = the category with Objects = Finite sets,

Schematic description of a category with $Ob(\mathcal{C}) = \{x, y\}$

$$\begin{tikzcd} x \arrow[loop left, "C_4"] \arrow[r, "C_4"] & y \arrow[loop right, "C_4"] \end{tikzcd}$$

4 morphisms $x \to y$, $End(x) = C_4$, $End(y) = C_4$

$\overset{\shortparallel}{Hom_{\mathcal{C}}(x,x)}$

Question:

How many morphisms does $F(Q)$ have when

$$Q = \begin{tikzcd} x \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & z \\ & y \arrow[ur, "\gamma"'] & \end{tikzcd}$$

A 3

B 4

C 5

D 6

E 7

F 8

G Infinitely many.

$$\begin{tikzcd} \bullet \arrow[loop left, "C_4"] \arrow[r, "C_4"] & \bullet \arrow[loop right, "C_4"] \end{tikzcd}$$

$\{1_y, a_y \ \dots\}$

$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$

$\{1, a, a^2, a^3\}$

Composition

$a^2 \circ a_x = a^3$

$a_x^2 \circ a_x = a_x^3$

Pre-class Warm-up!

How many morphisms are there in the free category generated by the quiver

$$\begin{tikzcd} w \arrow[r, "\alpha"] & x \arrow[r, "\beta"] & y & z \arrow[l, "\gamma"'] \end{tikzcd}$$

A 3

B 4

C 7

D 8

Constructions.

- The product of two categories $\mathcal{C}$, $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with objects (c, d), $c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$, morphisms $(c,d) \xrightarrow{(f,g)} (c', d')$ where $f : c \to c'$ in $\mathcal{C}$, $g : d \to d'$ in $\mathcal{D}$.

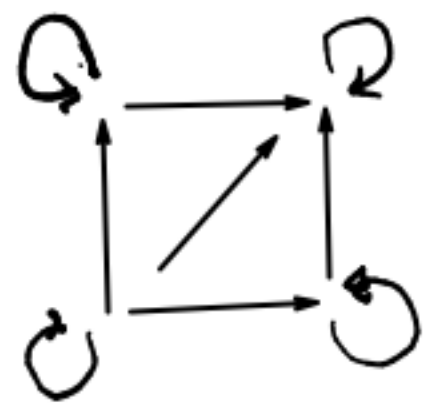
- If $\mathcal{C}$ is a category, the opposite category is $\mathcal{C}^{op}$ with $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and morphisms $\bar{\alpha}$ where α is a morphism in $\mathcal{C}$. If $\alpha : x \to y$ then $\bar{\alpha} : y \to x$. $\bar{\beta} : z \to y$, $\beta : y \to z$

$$\bar{\alpha}\bar{\beta} := \overline{\beta\alpha}$$

Question: Let I be the poset $0 \xrightarrow{\alpha} 1$

How many morphisms does $I \times I$ have?

A 1
B 2
C 4
D 6
E 8
F 9 ✓



Functors

Definition. $\mathcal{C}, \mathcal{D}$ are categories. the specification of

- $\forall\ x \in Ob(\mathcal{C})$, an object $T(x) \in Ob(\mathcal{D})$
- $\forall$ morphisms $f: x \to y$ in $\mathcal{C}$, a morphism $T(f): T(x) \to T(y)$ in $\mathcal{D}$

so that

1. $T(fg) = T(f)T(g)$
2. $T(1_x) = 1_{T(x)} \quad \forall x \in Ob(\mathcal{C})$

Question: did we need to put in $T(1) = 1$ always, or did it follow from the other axioms

Examples $F: AbGroups \to Groups$

Inclusion $F(A) = A$, $F(f) = f$

Forgetful functor like $F: Groups \to Set$

$F(G) = G$ regarded as a set

or $k[x]$-mod $\to$ k-mod = vector spaces over k.

If G and H are groups we get categories $\mathcal{G}, \mathcal{H}$. A functor $F: \mathcal{G} \to \mathcal{H}$ is 'the same thing as' a group homomorphism $G \to H$.

If P and Q are posets a functor $F: P \to Q$ is 'the same thing as' an order preserving map $P \to Q$.

If X is a set let $R(X)$ = free R-module with X as a basis.

R is a commutative ring

$R(-)$ is a functor $Set \to R\text{-mod}$

$F(-): Set \to Group$ is a functor. ($F(X)$ = free group generated by X)

If M is a right R-module, L is a left R-module, we have functors

$M \otimes - \; : R\text{-mod} \to \text{Ab Group}$

$\text{Hom}_R(L, -) : R\text{-mod} \to \text{Ab Group}$

R-mod = category of left R-modules

mod-R = right

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$

$\mathcal{C} \to \mathcal{D}$

A contravariant functor has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$.

If G is a group (or a monoid) a functor F : G -> R-mod

A repn of G over R is a homomorphism $G \to GL(V)$ for some R-module V.

$F : G \to R\text{-mod}$ is

$F(*) = V$

$\forall \; g : * \to *$, $F(g) : V \to V$ is an R-module homomorphism.

$F(g g^{-1}) = F(1_*) = 1_V : V \to V$

$= F(g) F(g^{-1})$

$F(g^{-1}) = F(g)^{-1}$

Definition. A category C is small if Ob(C) is a set.

Example:
SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$\rho : G \to GL(V)$ $\quad \sigma : G \to GL(W)$

$\rho \to \sigma$

Morphisms in the category Reps of G are R-linear maps $\theta : V \to W$ so that $\theta(\rho(g)(v)) = \sigma(g)(\theta(v))$.

i.e the square

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\theta"'] & V \arrow[d, "\theta"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

commutes $\forall g \in G$.

This defines a homomorphism of repns $\rho \to \sigma$

The functors between the category of representations of a group G and the category of RG-modules.

Reps of G are the same thing as RG-modules

RG = group ring of G
= free R-module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

$$\text{Reps of } G \rightleftarrows RG\text{-mod}$$

Given $\rho : G \to GL(V)$ get an RG-module V

$$\left(\sum_{g\in G} a_g g\right) \cdot v := \sum_{g \in G} a_g \, \rho(g)(v)$$

Pre-class Warm-up!

Let F : C -> D be a functor. Which of the following do you think means that F is an isomorphism of categories?

A For all objects x of C , x is isomorphic to F(x).

B For all pairs of objects x, y of C, F induces an isomorphism Hom_C(x,y) ≈ Hom_D(F(x),F(y))

C There is a functor G : D -> C so that FG = 1_D and GF = 1_C. ✓

D None of the above.

Example. The categorys

$$\text{Reps of } G \text{ over } R \underset{G}{\overset{F}{\rightleftarrows}} RG\text{-mod}$$

are isomorphic.

Given $\rho : G \to GL(V)$ we get an RG-module $F(\rho) = V$, $\sum a_g g \cdot v := \sum a_g \rho(g)(v)$

Given an RG-module W we get a homom. $G(W) : G \to GL(W)$
$g \mapsto (w \mapsto g \cdot w)$.

$GF(\rho) = \rho$

$FG(V) = F(G \to GL(V))$
$= V$ with original module action of RG.

If Q is a directed graph (= a quiver), a representation of Q over R is the specification of

- for each vertex x of Q, an R-module M_x
- for each arrow $x \xrightarrow{\alpha} y$ in Q an R-module homomorphism $M_x \xrightarrow{M(\alpha)} M_y$

Example: $Q = \alpha \circlearrowleft \cdot x$ a repn M is an R-module M_x with an R-linear map $M(\alpha): M_x \to M_x$.

This is the same thing as a $R[t]$-module where t acts via α.

It is "the same thing" as a functor $F(Q)$ -> R-mod

A homomorphism of quiver representations the specification of, for each vertex x of Q $= x \in \mathrm{Ob}\,F(Q)$ an R-module homomorphism

$$F_1(x) \xrightarrow{\mu_x} F_2(x)$$
$$\| \qquad\qquad \|$$
$$M_{1x} \qquad\qquad M_{2x}$$

so that every diagram commutes

$$\begin{tikzcd} F_1(x) \arrow[r, "\mu_x"] \arrow[d, "F_1(\alpha)"'] & F_2(x) \arrow[d, "F_2(\alpha)"] \\ F_1(y) \arrow[r, "\mu_y"] & F_2(y) \end{tikzcd}$$

($\forall$ morphisms α in $F(Q)$)

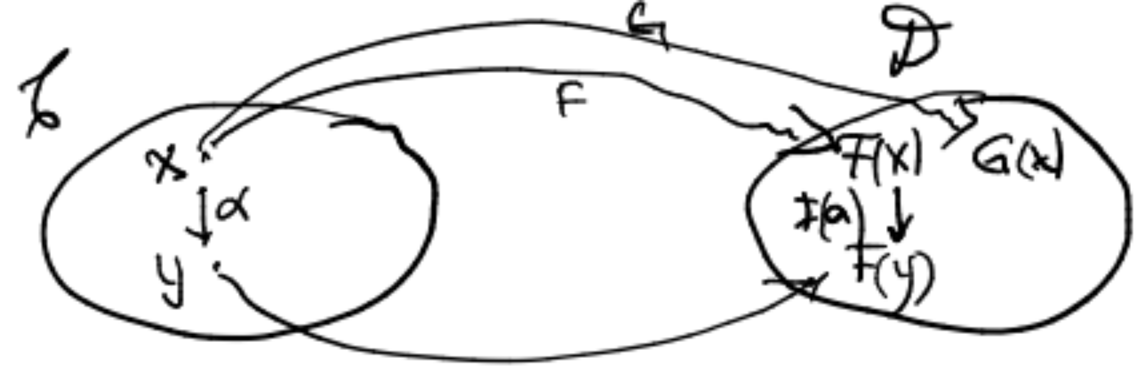
Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F , G : C -> D by functors. A natural transformation $\mu : F \to G$ is the specification of, for each object $x \in Ob(\mathcal{C})$ a morphism $\mu_x : F(x) \to G(x)$ in $\mathcal{D}$ so that, $\forall$ morphisms $\alpha : x \to y$ in $\mathcal{C}$ the square

$$\begin{tikzcd} F(x) \arrow[r, "\mu_x"] \arrow[d, "F(\alpha)"'] & G(x) \arrow[d, "G(\alpha)"] \\ F(y) \arrow[r, "\mu_y"] & G(y) \end{tikzcd}$$

commutes.



Examples: Homomorphism of group representations and of quiver

Regarding repns of G as functors $\rho : G \to R\text{-mod}$ a homomorphism $\rho \to \sigma$ is a natural transformation. $\rho \to \sigma$

Regarding repns of a quiver Q as functors $F(Q)$, a homomorphism of repns $F_1 \to F_2$ is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

SCat has $\left[\begin{array}{l} \text{Objects : categories} \\ \text{Morphisms : functors} \end{array}\right.$

Fun $\left[\begin{array}{l} \text{Objects : functors} \\ \text{Morphisms : nat transfos} \end{array}\right.$

In fact SCat is a 2-category.

Example: Hom(C,D) where C and D are categories.

Let $\mathcal{C}$, $\mathcal{D}$ be categories. We define a category $\mathrm{Hom}_{SCat}(\mathcal{C},\mathcal{D}) = \mathrm{Fun}(\mathcal{C},\mathcal{D})$

with objects := functors $\mathcal{C} \to \mathcal{D}$

morphisms $F \to G$:= natural transformations $F \to G$.

Implicit: we can compose natural transfos $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$

Then $(\lambda\mu)_x := \lambda_x \mu_x : F(x) \to H(x)$ in $\mathcal{D}$

There is a $1_F : F \to F$ for each functor F.

The double dual.

Let V be a finite dimensional vector space over a field k. Let $V^* = \mathrm{Hom}(V,k)$ be the vector space dual.

Question: Is the operation that sends V to V^*

A a functor?

B a natural transformation?

C Neither.

Natural isomorphism, equivalence of categories.