

Category Theory

Eisenbud's Appendix 5 has the right topics but is brief with a shortage of examples.

Definition. A category \mathcal{C} is the specification of

1. A class of things called 'objects' $x \in \text{Ob}(\mathcal{C})$ means x is an object of \mathcal{C} .
2. For each pair $x, y \in \text{Ob}(\mathcal{C})$ we have a set $\text{Hom}_{\mathcal{C}}(x, y)$ of things called morphisms.

3. A rule of composition

$$\text{Hom}(y, z) \times \text{Hom}(x, y) \longrightarrow \text{Hom}(x, z)$$

so that $(g, f) \longmapsto gf$ (or $g \circ f$)

- a. $(hg)f = h(gf)$ always, whenever it is defined.

- i. For all $x \in \text{Ob}(\mathcal{C})$, there exists a morphism $1_x : x \rightarrow x$ so that $f1_x = f \quad \forall f : x \rightarrow y, 1_x j = j \quad \forall j : w \rightarrow x$

Morphism notation If $f \in \text{Hom}_{\mathcal{C}}(x, y)$ we write $f : x \rightarrow y$ to denote this. x is the 'domain' of f , y is the 'codomain' or 'target' of f .

Examples

1. Set = category with objects = sets
morphisms = maps of sets.

Top = category of topological spaces
morphisms = continuous maps

Group: morphisms = group homomorphisms

R -mod: Objects are R -modules
Morphisms are R -module homomorphisms.

2. A poset P may be regarded as a category \mathcal{P} with $\text{Ob}(\mathcal{P}) = \text{elements of } P$, \exists unique morphism $x \rightarrow y \Rightarrow x \leq y$ in P .

Pre-class Warm-up!!!

Suppose $f : M \rightarrow N$ is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

- A. For all pairs of homomorphisms $g, h : L \rightarrow M$, if $fg = fh$ then $g = h$.
- B. For all pairs of homomorphisms $g, h : N \rightarrow Q$, if $gf = hf$ then $g = h$.
- C. Neither of the above.

$B \Leftrightarrow f$ is onto.

$$A. \quad L \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} M \xrightarrow{f} N \quad fg = fh \\ \Rightarrow g = h$$

Proposition $A \Leftrightarrow f$ is 1-1.

Proof "A \Rightarrow 1-1" If f is not 1-1 then $\ker f \neq 0$. Take

$L = \ker f$, $g : L \rightarrow M$ is inclusion, $h : L \rightarrow M$ is zero

Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 \Rightarrow A" 1-1 $\Leftrightarrow \ker f = 0$

If $fg = fh$ then $\forall x \in L$, $fg(x) = fh(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\Leftrightarrow \exists$ a morphism $g: y \rightarrow x$
so that $gf = 1_x$ and $fg = 1_y$.

The following is not equivalent to f
being an isomorphism.
or $\Leftrightarrow f$ is 1-1 and onto?
 \Downarrow
A, say f is a monomorphism
 \Uparrow
B, say f is an epimorphism

I just suggested
it for the purposes
of discussion.

More examples: a group, a monoid

Given a group G we may
construct a category \mathcal{G}
with only one object $*$
and where $\text{Hom}(*, *) = G$
composition: = multiplication
in G !

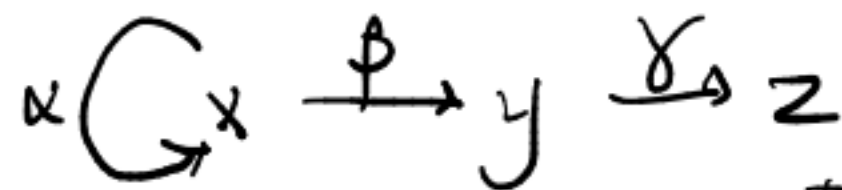
If M is a monoid we
construct a category \mathcal{M}
with one object $*$
 $\text{Hom}(*, *) = M$.

Question. Why do we take this definition of

More examples: weird categories.

Free categories

Let Q be a directed graph (quiver)



Construct a category $F(Q)$ where
 $Ob(F(Q)) = \text{vertices of } Q$

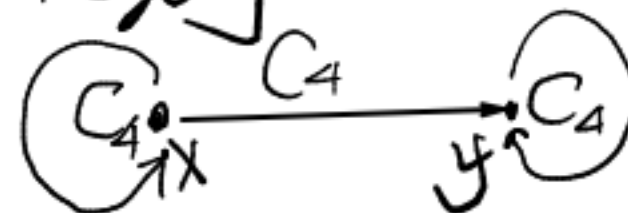
Morphisms = all possible words
 in the edges of Q where
 the end of a symbol = start of next.

Example $Ob = \{x, y, z\}$

Morphisms = $\{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \dots\}$

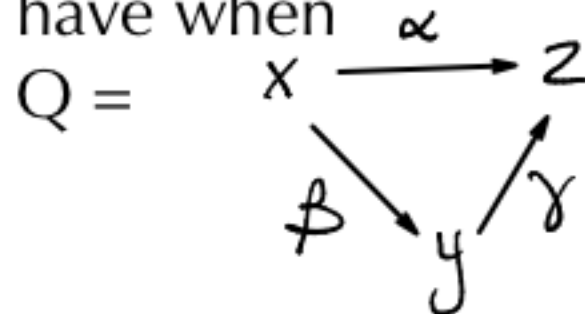
FI = the category with Objects = Finite sets,

Schematic description of
 a category with $Ob(e) = \{x, y\}$



4 morphisms $x \rightarrow y$, $End_x(x) = C_4$
 $End_y(y) = C_4$
 Question: $Hom_C(x, x)$

How many morphisms does $F(Q)$
 have when



- A 3
- B 4
- C 5
- D 6
- E 7
- F 8
- G Infinitely many.



$$C_A = \{1_x, a_x, a_x^2, a_x^3\}$$

$$\{1, a, a^2, a^3\}$$

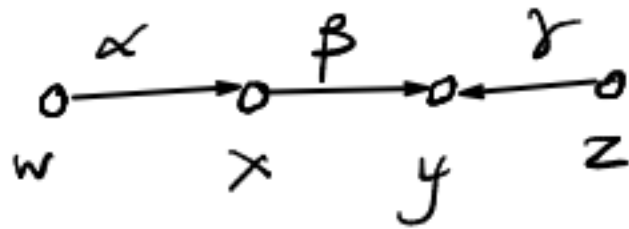
Composition

$$a^2 \circ a_x = a^3$$

$$a_x^2 \circ a_x = a_x^3$$

Pre-class Warm-up!

How many morphisms are there in the free category generated by the quiver



- A 3
- B 4
- C 7
- D 8

Constructions.

= The product of two categories \mathcal{C}, \mathcal{D} is the category

$\mathcal{C} \times \mathcal{D}$ with objects (c, d)

$c \in \text{Ob}(\mathcal{C}), d \in \text{Ob}(\mathcal{D})$

morphisms $(c, d) \xrightarrow{(f, g)} (c', d')$

where $f: c \rightarrow c'$ in $\mathcal{C}, g: d \rightarrow d'$ in \mathcal{D} .

= If \mathcal{C} is a category, the opposite category is \mathcal{C}^{op} with $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$

and morphisms $\bar{\alpha}$ where α is a morphism in \mathcal{C} . If $\alpha: x \rightarrow y$ then

$\bar{\alpha}: y \rightarrow x$. $\bar{\beta}: z \rightarrow y, \beta: y \rightarrow z$

$$\bar{\alpha} \bar{\beta} := \overline{\beta \alpha}$$

Question: Let I be the poset $0 \xrightarrow{\alpha} 1$

How many morphisms does $I \times I$ have?

A 1

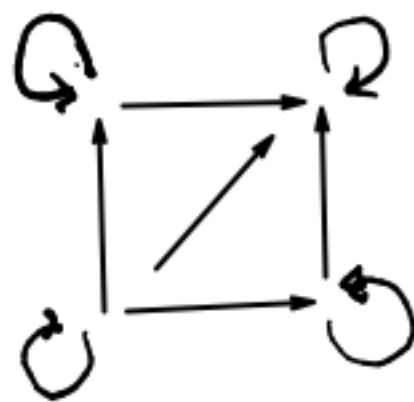
B 2

C 4

D 6

E 8

F 9 ✓



Functors

Definition. \mathcal{C}, \mathcal{D} are categories.
the specification

- $\forall x \in \text{Ob}(\mathcal{C})$, an object $T(x) \in \text{Ob}(\mathcal{D})$
- \forall morphisms $f: x \rightarrow y$ in \mathcal{C} ,
a morphism $T(f): T(x) \rightarrow T(y)$ in \mathcal{D}
so that

1. $T(fg) = T(f)T(g)$
2. $T(1_x) = 1_{T(x)} \quad \forall x \in \text{Ob}(\mathcal{C})$

Question: did we need to put in $T(1) = 1$ always, or did it follow from the other axioms

Examples $F: \text{AbGroups} \rightarrow \text{Groups}$
Inclusion $F(A) = A, F(f) = f$

Forgetful functor like $F: \text{Groups} \rightarrow \text{Set}$
 $F(G) = G$ regarded as a set
or $R[x]\text{-mod} \rightarrow R\text{-mod} = \text{vector spaces over } R$.

If G and H are groups
we get categories \mathcal{G}, \mathcal{H} . A functor
 $F: \mathcal{G} \rightarrow \mathcal{H}$ is 'the same thing as' a
group homomorphism $G \rightarrow H$.

If P and Q are posets a functor
 $F: P \rightarrow Q$ is 'the same thing as'
an order preserving map $P \rightarrow Q$.

If X is a set let $R(X) =$ free R -module
 R is a commutative ring with X as a basis.
 $R(-)$ is a functor $\text{Set} \rightarrow R\text{-mod}$

$F(-): \text{Set} \rightarrow \text{Group}$ is a functor.
free group generated by X

If M is a right R -module, L is a left R -module, we have functors

$$M \otimes - : R\text{-mod} \rightarrow \text{Ab Group}$$

$$\text{Hom}_R(L, -) : R\text{-mod} \rightarrow \text{Ab Group}$$

$R\text{-mod}$ = category of left R -modules
 $\text{mod-}R$ = right

These are covariant functors. The functor

$$\text{Hom}_R(-, L) : R\text{-mod} \rightarrow \text{Abelian groups}$$

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$

A contravariant functor $L \rightarrow \mathcal{D}$ has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant)

$$\text{functor } L^{\text{op}} \rightarrow \mathcal{D}$$

If G is a group (or a monoid) a functor $F : G \rightarrow R\text{-mod}$

A repr of G over R is a homomorphism $G \rightarrow GL(V)$ for some R -module V .

$F : G \rightarrow R\text{-mod}$ is

$$F(*) = V$$

$\forall g : * \rightarrow *$, $F(g) : V \rightarrow V$
 is an R -module homomorphism.

$$F(gg^{-1}) = F(1_*) = 1_V : V \rightarrow V$$

$$= F(g)F(g^{-1})$$

$$F(g^{-1}) = F(g)^{-1}$$

Definition. A category C is small if $\text{Ob}(C)$ is a set.

Example:

SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$$\rho: G \rightarrow \text{GL}(V) \quad \sigma: G \rightarrow \text{GL}(W)$$

$$\rho \rightarrow \sigma$$

Morphisms in the category $\text{Reps of } G$

are R -linear maps $\theta: V \rightarrow W$

$$\text{so that } \theta(\rho(g)(v)) = \sigma(g)(\theta(v))$$

i.e. the square

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \theta \downarrow & & \downarrow \theta \\ W & \xrightarrow{\sigma(g)} & W \end{array} \quad \text{commutes} \quad \forall g \in G.$$

This defines a homomorphism of reps $\rho \rightarrow \sigma$

The functors between the category of representations of a group G and the category of RG -modules.

Reps of G are the same thing as RG -modules

RG = group ring of G
= free R -module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

$$\text{Reps of } G \quad \rightleftarrows \quad RG\text{-mod}$$

Given $\rho: G \rightarrow \text{GL}(V)$

get an RG -module V
 $(\sum_{g \in G} a_g g) \cdot v := \sum_{g \in G} a_g \rho(g)(v)$

Pre-class Warm-up!

Let $F : C \rightarrow D$ be a functor. Which of the following do you think means that F is an isomorphism of categories?

- A For all objects x of C , x is isomorphic to $F(x)$.
- B For all pairs of objects x, y of C , F induces an isomorphism $\text{Hom}_C(x, y) \approx \text{Hom}_D(F(x), F(y))$
- C There is a functor $G : D \rightarrow C$ so that $FG = 1_D$ and $GF = 1_C$. ✓
- D None of the above.

Example. The categories

$\text{Reps of } G \text{ over } R \xrightleftharpoons[G]{F} RG\text{-mod}$

are isomorphic.

Given $\rho : G \rightarrow GL(V)$ we get an RG -module $F(\rho) = V := \sum a_g g \cdot v := \sum a_g \rho(g)(v)$

Given an RG -module W we get a homom. $G(W) : G \rightarrow GL(W)$
 $g \mapsto (w \mapsto g \cdot w)$.

$$GF(\rho) = \rho$$

$$FG(V) = F(G \rightarrow GL(V)) = V \text{ with original module action of } RG.$$

If Q is a directed graph (= a quiver), a representation of Q over R is the specification of

- for each vertex x of Q , an R -module M_x
- for each arrow $x \xrightarrow{\alpha} y$ in Q an R -module homomorphism $M_x \xrightarrow{M(\alpha)} M_y$

Example: $Q = \alpha \circlearrowleft x$ a repr M is an R -module M_x with an R -linear map $M(\alpha): M_x \rightarrow M_x$.
 This is the same thing as an $R[t]$ -module where t acts via α .

It is the same thing as a functor $F(Q) \rightarrow R\text{-mod}$
 A homomorphism of quiver representations

of, for each vertex x of $Q = x \in \text{Ob } F(Q)$ the specification an R -module homomorphism

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\mu_x} & F_2(x) \\ \parallel & & \parallel \\ M_{1,x} & & M_{2,x} \end{array}$$

so that every diagram commutes

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\mu_x} & F_2(x) \\ F_1(\alpha) \downarrow & & \downarrow F_2(\alpha) \\ F_1(y) & \xrightarrow{\mu_y} & F_2(y) \end{array}$$

(\forall morphisms α in $F(Q)$)

Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

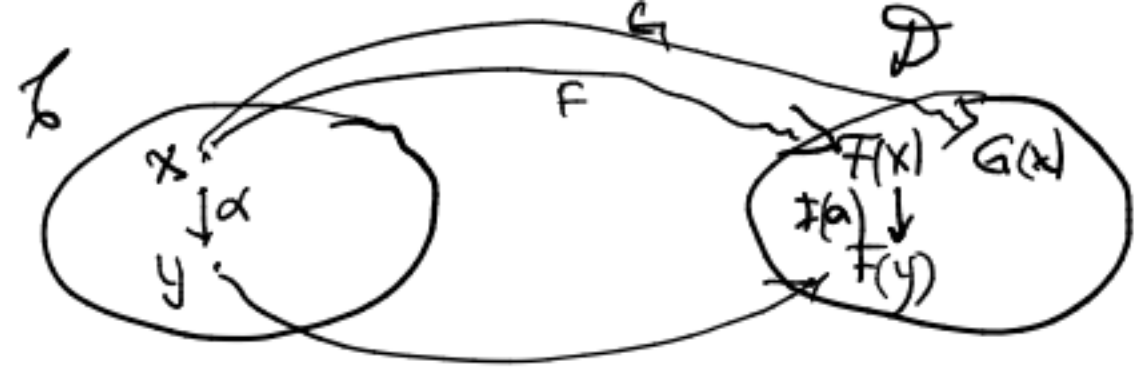
Definition. Let $F, G : C \rightarrow D$ be functors.

A natural transformation

$\mu : F \rightarrow G$ is the specification of, for each object $x \in \text{Ob}(C)$ a morphism $\mu_x : F(x) \rightarrow G(x)$ in D so that, \forall morphisms $\alpha : x \rightarrow y$ in C the square

$$\begin{array}{ccc} F(x) & \xrightarrow{\mu_x} & G(x) \\ F(\alpha) \downarrow & & G(\alpha) \downarrow \\ F(y) & \xrightarrow{\mu_y} & G(y) \end{array}$$

commutes.



Examples: Homomorphism of group representations and of quiver

Regarding reps of G as functors $\rho : G \rightarrow R\text{-mod}$ a homomorphism $\rho \rightarrow \sigma$ is a natural transformation.

Regarding reps of a quiver Q as functors $F(Q)$, a homomorphism of reps $F_1 \rightarrow F_2$ is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

$SCat$ has

- Objects: categories
- Morphisms: functors

Fun

- Objects: functors
- Morphisms: nat transfs

In fact $SCat$ is a 2-category.

Example: $Hom(C, D)$ where C and D are categories.

Let \mathcal{C}, \mathcal{D} be categories. We define a category $Hom_{SCat}(\mathcal{C}, \mathcal{D}) = Fun(\mathcal{C}, \mathcal{D})$ with objects: = functors $\mathcal{C} \rightarrow \mathcal{D}$

morphisms $F \rightarrow G$
:= natural transformations $F \rightarrow G$.

Implicit: we can compose natural transfs $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$
then $(\lambda\mu)_x := \lambda_x \mu_x : F(x) \rightarrow H(x)$
in \mathcal{D}

There is a $1_F : F \rightarrow F$ for each functor F .

Pre-class Warm-up!!

Let M be an R -module. Is the operation that sends

$\text{Hom}_R(R, M)$ to M

by sending

$$\phi \longmapsto \phi(1)$$

- A a functor
- B a natural transformation
- C a category
- D none of the above.

We have functors
 $\text{Hom}_R(R, -): R\text{-mod} \rightarrow R\text{-mod}$
 $1: R\text{-mod} \rightarrow R\text{-mod}.$

$$\eta: \text{Hom}(R, -) \rightarrow 1$$

specified by

$$\eta_M: \text{Hom}(R, M) \rightarrow 1(M) = M$$

$$\eta_M(\phi) = \phi(1).$$

is a natural transformation.

If $\alpha: M \rightarrow N$ then

$$\begin{array}{ccc} \text{Hom}(R, M) & \xrightarrow{\eta_M} & M \\ \downarrow \text{Hom}(R, \alpha) & & \downarrow \alpha \\ \text{Hom}(R, N) & \xrightarrow{\eta_N} & N \end{array}$$

commutes.

The double dual.

Let V be a finite dimensional vector space over a field k . Let $V^{\wedge*} = \text{Hom}(V, k)$ be the vector space dual.

Question: Is the operation that sends V to $V^{\wedge*}$

A a functor?

B a natural transformation?

$$\begin{array}{ccc} k\text{-mod} & \longrightarrow & k\text{-mod} \\ V & \longmapsto & V^* \end{array}$$

is a contravariant functor i.e.
a functor $k\text{-mod}^{\text{op}} \rightarrow k\text{-mod}$

$$F(V) = V^{**}$$

$$F: k\text{-mod} \rightarrow k\text{-mod},$$

$$V \neq V^{**}$$

We have natural transformation

$$\eta: 1 \rightarrow (-)^{**}$$

$$\eta_V: V \rightarrow V^{**}$$

$$\eta_V(w) = (f \mapsto f(w))$$

for $w \in V$.

Also there is a nat. transfn
 $\theta: (-)^{**} \rightarrow 1$ if V is
finite dimensional. On the category where
 $\dim V < \infty \Rightarrow 1$ and $(-)^{**}$ are
naturally isomorphic functors.

Question:

Is the functor $V \rightarrow V^*$ naturally isomorphic to the identity functor on finite-dimensional vector spaces?

A Yes

B No ✓

\uparrow is covariant: $k\text{-mod} \rightarrow k\text{-mod}$
 $(-)^*$ is contravariant.

Example.

Given a finite set X and a ring R we may construct the free R -module R^X on X . Let N be an R -module.

Consider the two functors $\text{Set} \rightarrow \text{Set}$ specified by
if Y is a set then

$$F(Y) = \text{Hom}_{\text{Set}}(Y, N)$$

$$G(Y) = \text{Hom}_R(R^Y, N)$$

Any map of sets $\phi: Y \rightarrow N$
extends uniquely to an
 R -module homomorphism

$$\eta_Y(\phi): R^Y \rightarrow N \quad \text{Here } \eta_Y$$

is a bijection of sets

$$\eta_Y: F(Y) \rightarrow G(Y)$$

η is a natural isomorphism $F \cong G$

Natural isomorphism, equivalence of categories.

If $F, G: \mathcal{C} \rightarrow \mathcal{D}$ are two functors a natural isomorphism is a natural transformation

$\eta: F \rightarrow G$ for which \exists a nat. transfn. $\theta: G \rightarrow F$ so that $\theta\eta = 1_F$, $\eta\theta = 1_G$.

(Exercise: $\Leftrightarrow \eta_x: F(x) \rightarrow G(x)$ is an isomorphism in \mathcal{D} , $\forall x \in \text{Ob}(\mathcal{C})$.)

Example.

Let \mathcal{C} be the category with two objects x and y , and with only the identity morphisms. Let k be a field.

We compare

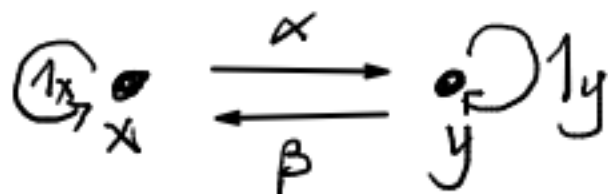
the category of functors $\mathcal{C} \rightarrow k\text{-mod}$ to the category of $k \times k$ -modules.

Example.

Let $C =$



Let $D =$



where $\beta\alpha = 1_x$ $\alpha\beta = 1_y$

Adjoint

Definition.

Let $F : C \rightarrow D$ and $G : D \rightarrow C$ be functors.

We say F is the left adjoint of G and G is the right adjoint of F if

Examples.

