Definition. A category C is the specification of

1. A class of things called 'objects'
   $x \in Ob(C)$ means x is an object of C.

2. For each pair $x, y \in Ob(C)$ we have a set $Hom_C(x,y)$ of things called morphisms.

3. A rule of composition
$$Hom(y,z) \times Hom(x,y) \longrightarrow Hom(x,z)$$
$$(g, f) \longmapsto gf \ (\text{or } g \circ f)$$

so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in Ob(C)$, there exists a morphism $1_x : x \to x$ so that
$$f 1_x = f \quad \forall f : x \to y, \quad 1_x j = j \ \forall j : w \to x$$

Morphism notation

If $f \in Hom_C(x,y)$ we write $f : x \to y$ to denote this.
x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples

1. Set = category with objects = sets, morphisms = maps of sets.

Top = category of topological spaces, morphisms = continuous maps

$Group$: morphisms = group homoms

R-mod: Objects are R-modules, Morphisms are R-module homomorphisms.

2. A poset P may be regarded as a category P with $Ob(P)$ = elements of P, $\exists$ unique morphism $x \to y$ $\Rightarrow x \leq y$ in P.

Pre-class Warm-up!!!

Suppose f : M -> N is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms g, h : L -> M, if fg = fh then g = h.

B. For all pairs of homomorphisms g, h : N -> Q, if gf = hf then g = h.

C. Neither of the above.

B $\iff$ f is onto.

A. $L \underset{h}{\overset{g}{\rightrightarrows}} M \xrightarrow{f} N \quad fg = fh \Rightarrow g = h$

Proposition A $\iff$ f is 1-1.

Proof "A $\Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero. Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A"   1-1 $\iff \ker f = 0$

If $fg = fh$ then $\forall x \in L$, $f g(x) = f h(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$$\Leftrightarrow \exists \text{ a morphism } g: y \to x$$

$$\text{so that } gf = 1_x \text{ and } fg = 1_y.$$

The following is not equivalent to f being an isomorphism.

or $\Leftrightarrow$ f is 1-1 and onto ?

$$\Updownarrow$$

A, say f is a monomorphism

I just suggested it for the purposes of discussion.

B say f is an epimorphism

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may construct a category $\mathcal{G}$ with only one object $*$ and where $\text{Hom}(*, *) = G$ and where composition := multiplication in G!
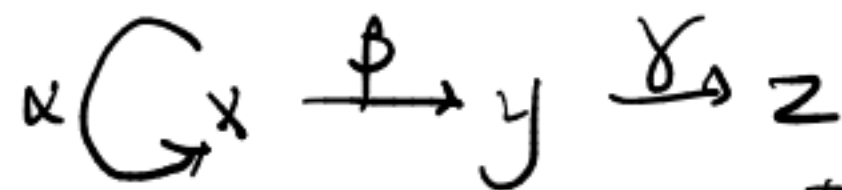
If M is a monoid we construct a category $\mathcal{M}$ with one object $*$ $\text{Hom}(*, *) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)

$$\alpha \circlearrowleft x \xrightarrow{\beta} y \xrightarrow{\gamma} z$$

Construct a category $F(Q)$ where
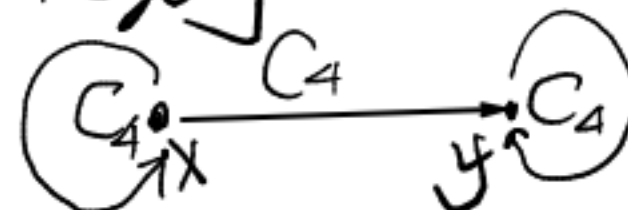$Ob(F(Q)) = $ vertices of Q
Morphisms = all possible words
    in the edges of Q where
the end of a symbol = start of next.
   Example $Ob = \{x, y, z\}$
Morphisms $= \{1_x, 1_y, 1_z, \alpha, \beta, \gamma,$
     $\alpha^2, \beta\alpha, \gamma\beta, \alpha^3 \beta\alpha^2, \gamma\beta\alpha$
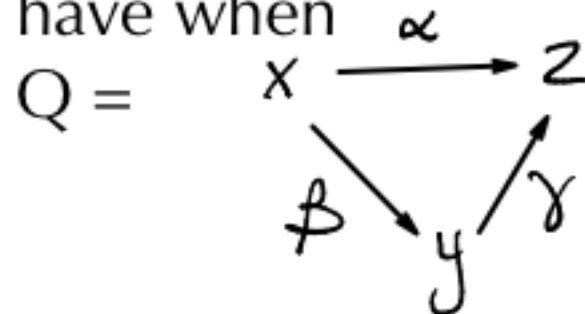 $\cdots \}$

FI = the category with Objects = Finite sets,

Schematic description of
a category with $Ob(\mathcal{C}) = \{x, y\}$

$$C_4 \circlearrowleft \underset{x}{\bullet} \xrightarrow{C_4} \underset{y}{\bullet} \circlearrowright C_4$$

4 morphisms $x \to y$, $End(x) = C_4$
$End(y) = C_4$
            $Hom_{\mathcal{C}}(x,x)$

Question:
How many morphisms does $F(Q)$
have when

$$Q = \quad x \xrightarrow{\alpha} z$$
$$\beta \searrow \quad \nearrow \gamma$$
$$y$$

A   3
B   4
C   5
D   6
E   7
F   8
G   Infinitely many.

C_4 $\xrightarrow{\;C_4\;}$ C_4 $\longleftarrow$ $\{1_y, a_y \cdots\}$

$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$

$\{1, a, a^2, a^3\}$

Composition $\qquad a^2 \circ a_x = a^3$

$\qquad\qquad\qquad\quad a_x^2 \circ a_x = a_x^3$

How many morphisms are there in the free category generated by the quiver

$$\underset{w}{\circ} \xrightarrow{\alpha} \underset{x}{\circ} \xrightarrow{\beta} \underset{y}{\circ} \xleftarrow{\gamma} \underset{z}{\circ}$$
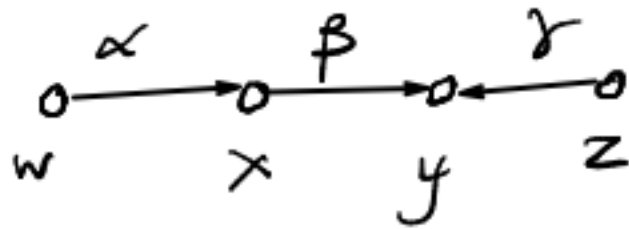
A 3

B 4

C 7

D 8

Constructions.

- The product of two categories $\mathcal{C}$, $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with objects (c,d)

$c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$

morphisms $(c,d) \xrightarrow{(f,g)} (c',d')$

where $f: c \to c'$ in $\mathcal{C}$, $g: d \to d'$ in $\mathcal{D}$.

- If $\mathcal{C}$ is a category, the opposite category is $\mathcal{C}^{op}$ with $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

and morphisms $\bar{\alpha}$ where α is a morphism in $\mathcal{C}$. If $\alpha: x \to y$ then

$\bar{\alpha}: y \to x$. $\bar{\beta}: z \to y$, $\beta: y \to z$

$\bar{\alpha}\bar{\beta} := \overline{\beta\alpha}$

Question: Let I be the poset $0 \xrightarrow{\alpha} 1$
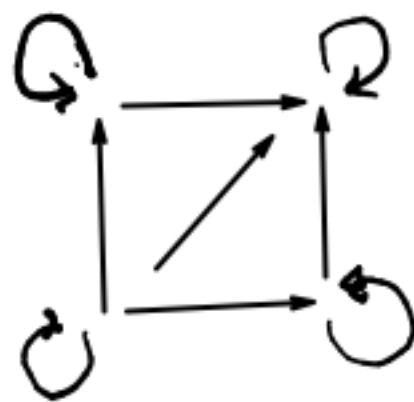
How many morphisms does $I \times I$ have?

A 1
B 2
C 4
D 6
E 8
F 9 ✓

Functors

Definition. $\mathcal{C}, \mathcal{D}$ are categories.
the specification
of
- $\forall\ x \in Ob(\mathcal{C})$, an object $T(x) \in Ob(\mathcal{D})$
- $\forall$ morphisms $f: x \to y$ in $\mathcal{C}$, a morphism $T(f): T(x) \to T(y)$ in $\mathcal{D}$
so that

1. $T(f\ g) = T(f)\ T(g)$

2. $T(1_x) = 1_{T(x)}$ $\qquad \forall x \in Ob(\mathcal{C})$

Examples $\quad F: AbGroups \to Groups$
Inclusion $\quad F(A) = A,\quad F(f) = f$

Forgetful functor like $F: Groups \to Set$
$F(G) = G$ regarded as a set
or $R[x]\text{-mod} \to R\text{-mod} = $ vector spaces over R.

If G and H are groups we get categories $\mathcal{G}, \mathcal{H}$. A functor $F: \mathcal{G} \to \mathcal{H}$ is 'the same thing as' a group homomorphism $G \to H$.

If P and Q are posets a functor $F: P \to Q$ is 'the same thing as' an order preserving map $P \to Q$.

If X is a set let $R(X) = $ free R-module. R is a commutative ring with X as a basis. $R(-)$ is a functor $Set \to R\text{-mod}$

$F(-): Set \to Group$ free group generated by X is a functor.

If M is a right R-module, L is a left R-module, we have functors

$$M \otimes - \;:\; R\text{-mod} \to Ab \; Group$$

$$\operatorname{Hom}_R(L, -) : R\text{-mod} \to Ab \; Group .$$

$$R\text{-mod} = \text{category of left } R\text{-modules}$$

$$\text{mod-}R = \qquad\qquad\qquad\text{right}$$

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$
$\;_{\mathscr{C}\to\mathscr{D}}$

A contravariant functor has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor $\mathscr{C}^{op} \to \mathscr{D}$.

If G is a group (or a monoid) a functor
F : G -> R-mod

A repn of G over R is a homomorphism $G \to GL(V)$ for some R-module V.

$F : G \to R\text{-mod}$ is
$F(*) = V$
$\forall \; g : * \to * , \quad F(g) : V \to V$
is an R-module homomorphism.
$$F(g \, g^{-1}) = F(1_*) = 1_V : V \to V$$
$$= F(g) F(g^{-1})$$
$$F(g^{-1}) = F(g)^{-1}$$

Definition. A category C is small if Ob(C) is a set.

Example:

SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$$\rho: G \to GL(v) \qquad \sigma: G \to GL(w)$$

$$\rho \longrightarrow \sigma$$

Morphisms in the category Reps of G are R-linear maps $\theta: V \to W$

so that $\theta(\rho(g)(v)) = \sigma(g)(\theta(v))$.

i.e the square

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \theta \downarrow & & \downarrow \theta \\ W & \xrightarrow{\sigma(g)} & W \end{array} \qquad \text{commutes} \quad \forall g \in G.$$

This defines a homomorphism of reps $\rho \to \sigma$

The functors between the category of representations of a group G and the category of RG-modules.

Reps of G are the same thing as RG-modules

RG = group ring of G
= free R-module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

$$\text{Reps of G} \rightleftarrows \text{RG-mod}$$

Given $\rho: G \to GL(v)$ get an RG-module V

$$\left(\sum_{g \in G} a_g g \right) \cdot v := \sum_{g \in G} a_g \rho(g)(v)$$

Let $F : C \to D$ be a functor. Which of the following do you think means that F is an isomorphism of categories?

A For all objects x of C, x is isomorphic to $F(x)$.

B For all pairs of objects x, y of C, F induces an isomorphism
$\text{Hom}_C(x,y) \approx \text{Hom}_D(F(x),F(y))$

C There is a functor $G : D \to C$ so that $FG = 1\_D$ and $GF = 1\_C$. ✓

D None of the above.

Example. The categorys.

$$\text{Reps of } G \text{ over } R \underset{G}{\overset{F}{\rightleftarrows}} RG\text{-mod}$$

are isomorphic.

Given $\rho : G \to GL(V)$ we get an RG-module
$$F(\rho) = V \ni \{ a_g g \cdot v := \sum a_g \rho(g)(v)$$

Given an RG-module W we get a homom. $G(W) : G \to GL(w)$
$$g \mapsto (w \mapsto g \cdot v).$$

$$GF(\rho) = \rho$$
$$FG(V) = F(G \to GL(V))$$
$$= V \text{ with original module action of } RG.$$

If Q is a directed graph (= a quiver), a representation of Q over R is the specification of

- for each vertex x of Q, an R-module M_x

- for each arrow $x \xrightarrow{\alpha} y$ in Q an R-module homomorphism

$$M_x \xrightarrow{M(\alpha)} M_y$$

Example: $Q = \alpha \circlearrowright \bullet x$ a repn M is an R-module M_x with an R-linear map $M(\alpha): M_x \to M_x$.

This is the same thing as a $R[t]$-module where t acts via α.

It is the same thing as a functor $F(Q) \to R\text{-mod}$

A homomorphism of quiver representations the specification of, for each vertex x of $Q = x \in Ob\, F(Q)$ an R-module homomorphism

$$F_1(x) \xrightarrow{\mu_x} F_2(x)$$

$$\| \qquad \qquad \|$$

$$M_{1\,x} \qquad \qquad M_{2\,x}$$

so that every diagram commutes

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\mu_x} & F_2(x) \\ \downarrow{\scriptstyle F_1(\alpha)} & & \downarrow{\scriptstyle F_2(\alpha)} \\ F_1(y) & \xrightarrow{\mu_y} & F_2(y) \end{array}$$

$(\forall$ morphisms α in $F(Q))$

Natural transformations

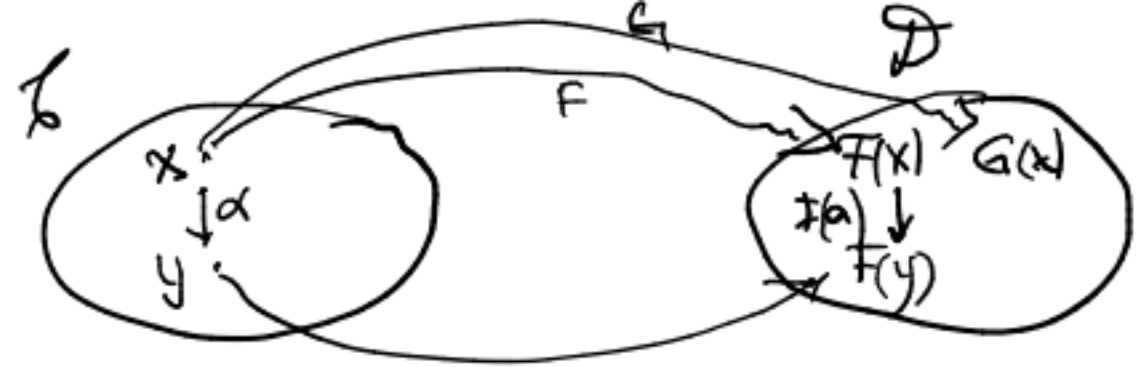These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F, $G : C \to D$ be functors.
A natural transformation

$\mu : F \to G$ is the specification

of, for each object $x \in Ob(C)$

a morphism $\mu_x : F(x) \to G(x)$ in D

so that, $\forall$ morphisms $\alpha : x \to y$

in C the square

$$F(x) \xrightarrow{\mu_x} G(x)$$
$$F(\alpha) \downarrow \qquad\quad G(\alpha) \downarrow$$
$$F(y) \xrightarrow{\mu_y} G(y)$$

commutes.

Examples: Homomorphism of group representations and of quiver

Regarding repns of G as functors

$\rho : G \to R\text{-mod}$ a homomorphism

$\rho \to \sigma$ is a natural transformation $\rho \to \sigma$

Regarding repns of a quiver Q as functors $F(Q)$, a homomorphism of repns $F_1 \to F_2$ is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

SCat has
- Objects : categories
- Morphisms : functors

Fun
- Objects : functors
- Morphisms : nat transf

In fact SCat is a 2-category.

Example: $\text{Hom}(C,D)$ where C and D are categories.

Let $\mathcal{C}, \mathcal{D}$ be categories. We define a category $\text{Hom}_{\text{SCat}}(\mathcal{C}, \mathcal{D})$
$$= \text{Fun}(\mathcal{C}, \mathcal{D})$$
with objects := functors $\mathcal{C} \to \mathcal{D}$

morphisms $F \to G$
:= natural transformation $F \to G$.

Implicit: we can compose natural transfs $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$ then $(\lambda \mu)_x := \lambda_x \mu_x : F(x) \to H(x)$ in $\mathcal{D}$

There is a $1_F : F \to F$ for each functor F.

Let M be an R-module. Is the operation that sends

$$\text{Hom}_R(R, M) \quad \text{to} \quad M$$

by sending

$$\phi \longmapsto \phi(1)$$

A   a functor

B   a natural transformation

C   a category

D   none of the above.

We have functors

$$\text{Hom}_R(R, -) : R\text{-mod} \longrightarrow R\text{-mod}$$
$$1 \quad\quad : R\text{-mod} \longrightarrow R\text{-mod}.$$

$$\eta : \text{Hom}(R, -) \longrightarrow 1$$

specified by

$$\eta_M : \text{Hom}(R, M) \longrightarrow 1(M) = M$$

$$\eta_M(\phi) = \phi(1).$$

is a natural transformation.

If $\alpha : M \to N$ then

$$
\begin{array}{ccc}
\text{Hom}(R, M) & \xrightarrow{\eta_M} & M \\
\downarrow{\scriptstyle\text{Hom}_R(R, \alpha)} & & \downarrow{\scriptstyle\alpha} \\
\text{Hom}(R, N) & \xrightarrow{\eta_N} & N
\end{array}
$$

commutes.

The double dual.

Let V be a finite dimensional vector space over a field k. Let $V^* = \text{Hom}(V,k)$ be the vector space dual.

Question: Is the operation that sends V to V^*

A a functor?

B a natural transformation?

$$k\text{-mod} \longrightarrow k\text{-mod}$$
$$V \longmapsto V^*$$

is a contravariant functor i.e. a functor $k\text{-mod}^{op} \longrightarrow k\text{-mod}$

$F(V) = V^{**}$

$F : k\text{-mod} \to k\text{-mod}$,

$V \not\cong V^{**}$

We have natural transformation

$\eta : 1 \longrightarrow (-)^{**}$

$\eta_V : V \longrightarrow V^{**}$

$\eta_V(w) = (f \longmapsto f(w))$
    for $w \in V$.

Also there is a nat. transfn

$\theta : (-)^{**} \longrightarrow V$ if V is finite dimensional. On the category where $\dim V < \infty \Rightarrow 1$ and $(-)^{**}$ are naturally isomorphic functors.

Question:

Is the functor $V \to V^*$ naturally isomorphic to the identity functor on finite-dimensional vector spaces?

A Yes

B No ✓

1 is covariant : $k\text{-mod} \to k\text{-mod}$

$(-)^*$ is contravariant.

Example.

Given a finite set X and a ring R we may construct the free R-module R^X on X. Let N be an R-module.

Consider the two functors $\text{Set} \to \text{Set}$ specified by

If Y is a set then

$$F(Y) = \text{Hom}_{\text{Set}}(Y, N)$$

$$G(Y) = \text{Hom}_R\left(R^Y, N\right).$$

Any map of sets $\phi : Y \to N$ extends uniquely to an R-module homomorphism

$$\eta_Y(\phi) : R^Y \to N \quad . \quad \text{Here } \eta_Y$$

is a bijection of sets

$$\eta_Y : F(Y) \to G(Y).$$

η is a natural isomorphism $F \cong G$

Pre-class Warm-up

Given a finite set X and a ring R we may construct the free R-module R^X on X.
Let f : M -> N be a homomorphism of R-modules.
Fixing X, consider the two functors R-mod -> Set specified by

$F(M) = \text{Hom}_{Set}(X, M)$

$G(M) = \text{Hom}_{R\text{-mod}}(R^X, M)$

On Wednesday we considered a bijection of sets
$\theta_M : F(M) \to G(M)$.

Does the square commute?

$$
\begin{array}{ccc}
F(M) & \xrightarrow{F(f)} & F(N) \\
\theta_M \uparrow & & \theta_N \uparrow \\
G(M) & \xrightarrow{G(f)} & G(N)
\end{array}
$$

A   Yes

If $\alpha : X \to M$ is a map of sets then
$F(f)(\alpha) := = f\alpha$.
$G(f)$ is also postcomposition with f

Natural isomorphism, equivalence of categories.

If $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are two functors a natural isomorphism is a natural transformation $\eta : F \to G$ for which $\exists$ a nat. transfn. $\theta : G \to F$ so that $\theta\eta = 1_F$, $\eta\theta = 1_G$.

$\left(\text{Exercise: } \iff \eta_x : F(x) \to G(x) \text{ is an isomorphism in } \mathcal{D}, \forall x \in Ob(\mathcal{C}).\right)$

Definition: Categories $\mathcal{C}, \mathcal{D}$ are equivalent $\iff \exists$ functors $F : \mathcal{C} \to \mathcal{D}$ $G : \mathcal{D} \to \mathcal{C}$ plus natural isomorphisms $\theta : GF \xrightarrow{\sim} 1_{\mathcal{C}}$ and $\psi : FG \xrightarrow{\sim} 1_{\mathcal{D}}$.

if $\mathcal{C}, \mathcal{D}$ are isomorphic as categories they are equivalent

Example.
Let C be the category with two objects x and y, and with only the identity morphisms. Let k be a field.
We compare
the category of functors $C \to k\text{-mod}$
to the category of $k \times k$-modules.

$\mathcal{C}$: $\overset{\circlearrowright 1_x}{\underset{x}{\bullet}} \qquad \overset{\circlearrowright 1_y}{\underset{y}{\bullet}}$

A functor $\mathcal{C} \to k\text{-mod}$ is the specification of two k-vector spaces

Get a functor $F : Fun(\mathcal{C}, k\text{-mod}) \to k \times k\text{-mod}$
$k \times k = \{(u, v) \mid u, v \in k\}$
$F(A) = $ the $k \times k$-module $A(x) \oplus A(y)$
   where $(u, v) \cdot (a, b) := (ua, vb)$

We define $G : k \times k\text{-mod} \to Fun(\mathcal{C}, k\text{-mod})$
$G(B) = \begin{cases} x \longmapsto (1, 0) \cdot B \\ y \longmapsto (0, 1) \cdot B \end{cases}$

$$F(A) = A(x) \oplus A(y) \in k \times k\text{-mod}$$

$$G(B) = \begin{cases} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{cases}$$
$$\in \text{Fun}(6, k\text{-mod}).$$

$$F: \text{Fun}(6, k) \leftrightarrows k \times k\text{-mod} : G$$

$$GF(A) = G\left(A(x) \oplus A(y)\right)$$
$$= \begin{cases} x \longmapsto (1,0)\left(A(x) \oplus A(y)\right) \\ \qquad \left(\ddot{A}(x) \oplus 0\right) \\ y \longmapsto 0 \oplus A(y). \end{cases}$$

looks like A but is different.

$$FG(B) = F\begin{pmatrix} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{pmatrix}$$
$$= (1,0)B \oplus (0,1)B.$$
$$= \text{set of pairs } ((1,0)a, (0,1)b)$$
$$a, b \in B.$$

looks like B but is different

F, G are not inverse isomorphisms

However $GF \not\simeq 1_{\text{Fun}(6, k\text{-mod})}$

$$FG \simeq 1_{k \times k\text{-mod}}$$

Define $\Theta: 1_{\text{Fun}()} \longrightarrow GF$

$\Theta_A: A \to GF(A)$ is morphism
of functors = a nat'l transfn.

$\Theta_{A,x}: A(x) \longrightarrow GF(A)(x) = (1,0)(A(x) \oplus A(y))$
$$a \longmapsto (a, 0) \in A(x) \oplus A(y)$$

Similarly we define $\psi: 1_{k \times k\text{-mod}} \to FG$.

$\text{Fun}(G, k\text{-mod})$ and

$k \times k\text{-mod}$

might be isomorphic categories.

It's a job to show this (if true)

However the categories are
equivalent. This is easier to show.

Speak of two categories being
equivalent rather than isomorphic.

Reprns of $\quad \bullet \quad \bullet \quad = Q$

'are the same as'

$k \times k$ —modules.

$F(Q) = 1_X \cdot G_x \qquad \bullet_y \mathcal{Q} 1_y$

Example.

Let $C = $ x $\circlearrowright 1_x$

Let $D = $ $1_x \circlearrowright x \underset{\beta}{\overset{\alpha}{\rightleftarrows}} y \circlearrowright 1_y$
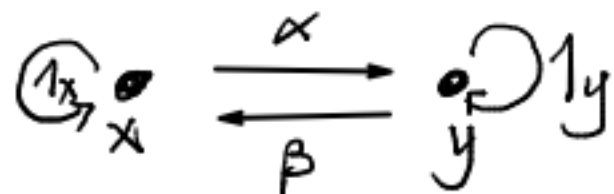
where $\beta\alpha = 1_x$ $\alpha\beta = 1_y$

We show $C \cong D$ are equivalent
but $C \not\cong D$ (not isomorphic).

We have

$\text{inc} : C \longrightarrow D$

$\qquad x \longmapsto x$

$\qquad 1_x \longmapsto 1_x$

and

$F : D \longrightarrow C$

$\qquad x \longmapsto x$

$\qquad y \longmapsto x$
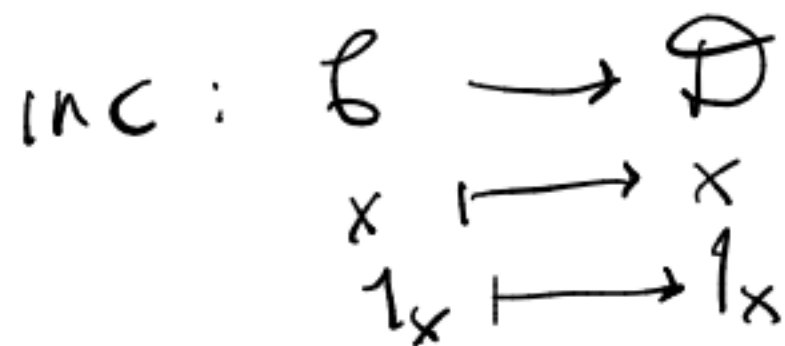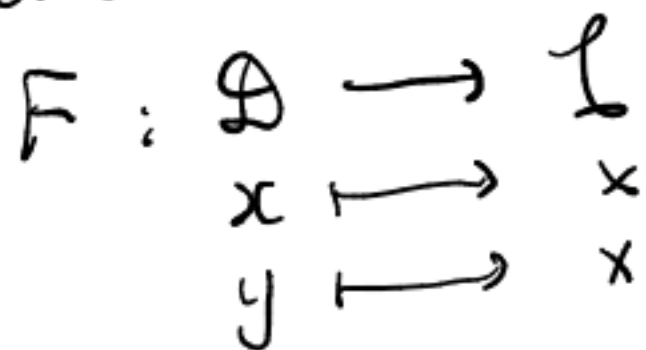
$1_x, 1_y, \alpha, \beta \longmapsto 1_x$

$F \circ \text{inc} = 1_C$.

$\text{inc} \circ F \neq 1_D$.

Define $\theta : 1_D \longrightarrow \text{inc} \circ F$

$\theta_x : 1_D(x) = x \longmapsto \text{inc} \circ F(x) = x$

$\theta_y : \qquad y \longmapsto \qquad x$

$\theta_x := 1_x \qquad \theta_y := \alpha$

Check this a nat'l. transformation.
It is iso ($1_x, \alpha$ are isomorphisms)

$C \cong D$.

Adjoints

Definition.
Let F : C -> D and G : D -> C be functors.
We say F is the left adjoint of G and G is
the right adjoint of F if

Examples.

The unit and counit of an adjunct ion.