Category Theory

Eisenbud's Appendix 5 has the right topics but is brief with a shortage of examples.

Definition. A category C is the specification of

1. A class of things called 'objects'
   $x \in Ob(\mathcal{C})$ means x is an object of $\mathcal{C}$.
2. For each pair $x, y \in Ob(\mathcal{C})$ we have a set $Hom_{\mathcal{C}}(x,y)$ of things called morphisms.
3. A rule of composition

$$Hom(y,z) \times Hom(x,y) \to Hom(x,z)$$
$$(g\ ,\ f) \longmapsto gf \ (\text{or } g \circ f)$$

so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in Ob(\mathcal{C})$, there exists a morphism $1_x : x \to x$ so that $f 1_x = f \ \forall f : x \to y$, $1_x j = j \ \forall j : w \to x$

Morphism notation If $f \in Hom(x,y)$ we write $f : x \to y$ to denote this. x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples

1. Set = category with objects = sets, morphisms = maps of sets.

Top = category of topological spaces, morphisms = continuous maps

Group: morphisms = group homoms

R-mod: Objects are R-modules, Morphisms are R-module homomorphisms.

2. A poset P may be regarded as a category $\mathcal{P}$ with $Ob(\mathcal{P})$ = elements of P, $\exists$ unique morphism $x \to y$ $\Rightarrow x \le y$ in P.

Pre-class Warm-up!!!

Suppose f : M -> N is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms g, h : L -> M, if fg = fh then g = h.

B. For all pairs of homomorphisms g, h : N -> Q, if gf = hf then g = h.

C. Neither of the above.

B $\Leftrightarrow$ f is onto.

A. $L \overset{g}{\underset{h}{\rightrightarrows}} M \xrightarrow{f} N$ $fg = fh \Rightarrow g = h$

Proposition A $\Leftrightarrow$ f is 1-1.

Proof "A $\Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero. Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A" 1-1 $\Leftrightarrow$ $\ker f = 0$

If $fg = fh$ then $\forall x \in L$, $fg(x) = fh(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\iff \exists$ a morphism $g : Y \to X$ so that $gf = 1_X$ and $fg = 1_Y$.

The following is not equivalent to f being an isomorphism.

or $\iff$ f is 1-1 and onto ?

$\Updownarrow$

A, say 'f is a monomorphism'

I just suggested it for the purposes of discussion.

B say 'f is an epimorphism'

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may construct a category $\mathcal{G}$ with only one object $*$ and where $\mathrm{Hom}(*,*) = G$

composition := multiplication in G!

If M is a monoid we construct a category $\mathcal{M}$ with one object $*$

$\mathrm{Hom}(*,*) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)

$$\alpha \circlearrowright x \xrightarrow{\beta} y \xrightarrow{\gamma} z$$

Construct a category $F(Q)$ where

$Ob(F(Q))$ = vertices of Q

Morphisms = all possible words in the edges of Q where the end of a symbol = start of next.

Example $Ob = \{x, y, z\}$

Morphisms $= \{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \ldots\}$

FI = the category with Objects = Finite sets,

Schematic description of a category with $Ob(\mathcal{C}) = \{x, y\}$

$$\begin{tikzcd} x \arrow[loop left, "C_4"] \arrow[r, "C_4"] & y \arrow[loop right, "C_4"] \end{tikzcd}$$

4 morphisms $x \to y$, $End(x) = C_4$, $End(y) = C_4$

$End(x) = Hom_{\mathcal{C}}(x,x)$

Question:

How many morphisms does $F(Q)$ have when

$$Q = \begin{tikzcd} x \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & z \\ & y \arrow[ur, "\gamma"'] & \end{tikzcd}$$

A 3

B 4

C 5

D 6

E 7

F 8

G Infinitely many.

$$\begin{tikzcd} \bullet \arrow[loop left, "C_4"] \arrow[r, "C_4"] & \bullet \arrow[loop right, "C_4"] \end{tikzcd}$$

$\{1_y, a_y \ldots\}$

$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$

$\{1, a, a^2, a^3\}$

Composition

$a^2 \circ a_x = a^3$

$a_x^2 \circ a_x = a_x^3$

Pre-class Warm-up!

How many morphisms are there in the free category generated by the quiver

$$w \xrightarrow{\alpha} x \xrightarrow{\beta} y \xleftarrow{\gamma} z$$

A 3

B 4

C 7

D 8

Constructions.

- The product of two categories $\mathcal{C}$, $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with objects (c,d), $c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$, morphisms $(c,d) \xrightarrow{(f,g)} (c',d')$ where $f: c \to c'$ in $\mathcal{C}$, $g: d \to d'$ in $\mathcal{D}$.

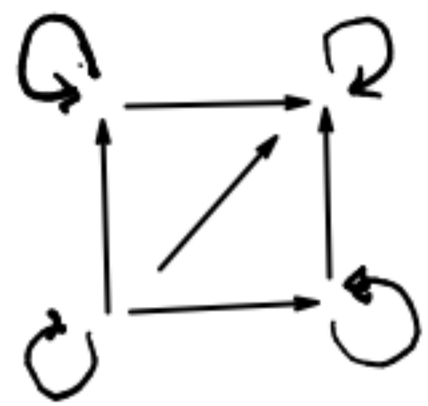
- If $\mathcal{C}$ is a category, the opposite category is $\mathcal{C}^{op}$ with $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and morphisms $\bar{\alpha}$ where α is a morphism in $\mathcal{C}$. If $\alpha: x \to y$ then $\bar{\alpha}: y \to x$. $\bar{\beta}: z \to y$, $\beta: y \to z$

$$\bar{\alpha}\bar{\beta} := \overline{\beta\alpha}$$

Question: Let I be the poset $0 \xrightarrow{\alpha} 1$

How many morphisms does I x I have?

A 1
B 2
C 4
D 6
E 8
F 9 ✓



Functors

Definition. $\mathcal{C}, \mathcal{D}$ are categories. the specification of

- $\forall\, x \in Ob(\mathcal{C})$, an object $T(x) \in Ob(\mathcal{D})$
- $\forall$ morphisms $f : x \to y$ in $\mathcal{C}$, a morphism $T(f) : T(x) \to T(y)$ in $\mathcal{D}$

so that

1. $T(fg) = T(f)\,T(g)$
2. $T(1_x) = 1_{T(x)} \quad \forall\, x \in Ob(\mathcal{C})$

Question: did we need to put in $T(1) = 1$ always, or did it follow from the other axioms

Examples $F : \text{AbGroups} \to \text{Groups}$

Inclusion $F(A) = A$, $F(f) = f$

Forgetful functor like $F : \text{Groups} \to \text{Set}$

$F(G) = G$ regarded as a set

or $k[x]\text{-mod} \to k\text{-mod} =$ vector spaces over k.

If G and H are groups we get categories $\mathcal{G}, \mathcal{H}$. A functor $F : \mathcal{G} \to \mathcal{H}$ is 'the same thing as' a group homomorphism $G \to H$.

If P and Q are posets a functor $F : P \to Q$ is 'the same thing as' an order preserving map $P \to Q$.

If X is a set let $R(X) =$ free R-module with X as a basis.

R is a commutative ring

$R(-)$ is a functor $\text{Set} \to R\text{-mod}$

$F(-) : \text{Set} \to \text{Group}$ is a functor. $F(X) =$ free group generated by X

If M is a right R-module, L is a left R-module, we have functors

$M \otimes - \; : R\text{-mod} \to \text{Ab Group}$

$\text{Hom}_R(L, -) : R\text{-mod} \to \text{Ab Group}$

R-mod = category of left R-modules
mod-R = right

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$
$\mathcal{C} \to \mathcal{D}$

A contravariant functor has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$.

If G is a group (or a monoid) a functor
F : G -> R-mod

A repn of G over R is a homomorphism $G \to GL(V)$ for some R-module V.

$F : G \to R\text{-mod}$ is

$F(*) = V$

$\forall \; g : * \to *, \; F(g) : V \to V$ is an R-module homomorphism.

$F(g g^{-1}) = F(1_*) = 1_V : V \to V$

$= F(g) F(g^{-1})$

$F(g^{-1}) = F(g)^{-1}$

Definition. A category C is small if Ob(C) is a set.

Example:
SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$\rho : G \to GL(V) \quad \sigma : G \to GL(W)$

$\rho \to \sigma$

Morphisms in the category Reps of G are R-linear maps $\theta : V \to W$ so that $\theta(\rho(g)(v)) = \sigma(g)(\theta(v))$.

i.e the square

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\theta"'] & V \arrow[d, "\theta"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

commutes $\forall g \in G$.

This defines a homomorphism of repns $\rho \to \sigma$

The functors between the category of representations of a group G and the category of RG-modules.

Reps of G are the same thing as RG-modules

RG = group ring of G
= free R-module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

$$\text{Reps of } G \rightleftarrows RG\text{-mod}$$

Given $\rho : G \to GL(V)$ get an RG-module V

$$\Big(\sum_{g \in G} a_g g\Big) \cdot v := \sum_{g \in G} a_g \rho(g)(v)$$

Pre-class Warm-up!

Let F : C -> D be a functor. Which of the following do you think means that F is an isomorphism of categories?

A For all objects x of C , x is isomorphic to F(x).

B For all pairs of objects x, y of C, F induces an isomorphism Hom_C(x,y) ≈ Hom_D(F(x),F(y))

C There is a functor G : D -> C so that FG = 1_D and GF = 1_C. ✓

D None of the above.

Example. The categorys

$$\text{Reps of } G \text{ over } R \underset{G}{\overset{F}{\rightleftarrows}} RG\text{-mod}$$

are isomorphic.

Given $\rho : G \to GL(V)$ we get an RG-module $F(\rho) = V$, $\sum a_g g \cdot v := \sum a_g \rho(g)(v)$

Given an RG-module W we get a homom. $G(W) : G \to GL(W)$
$g \longmapsto (w \mapsto g \cdot w)$.

$GF(\rho) = \rho$

$FG(V) = F(G \to GL(V))$
$= V$ with original module action of RG.

If Q is a directed graph (= a quiver), a representation of Q over R is the specification of

- for each vertex x of Q, an R-module M_x
- for each arrow $x \xrightarrow{\alpha} y$ in Q an R-module homomorphism

$$M_x \xrightarrow{M(\alpha)} M_y$$

Example: $Q = \begin{tikzcd} \bullet x \arrow["\alpha"', loop left] \end{tikzcd}$ a repn M is an R-module M_x with an R-linear map $M(\alpha) : M_x \to M_x$.

This is the same thing as a $R[t]$-module where t acts via α.

It is the same thing as a functor F(Q) -> R-mod

A homomorphism of quiver representations the specification of, for each vertex x of Q $= x \in \mathrm{Ob}\, F(Q)$ an R-module homomorphism

$$\begin{tikzcd} F_1(x) \arrow[r, "\mu_x"] \arrow[d, phantom, "\shortparallel"] & F_2(x) \arrow[d, phantom, "\shortparallel"] \\ M_{1x} & M_{2x} \end{tikzcd}$$

so that every diagram commutes

$$\begin{tikzcd} F_1(x) \arrow[r, "\mu_x"] \arrow[d, "F_1(\alpha)"'] & F_2(x) \arrow[d, "F_2(\alpha)"] \\ F_1(y) \arrow[r, "\mu_y"] & F_2(y) \end{tikzcd}$$

($\forall$ morphisms α in $F(Q)$)

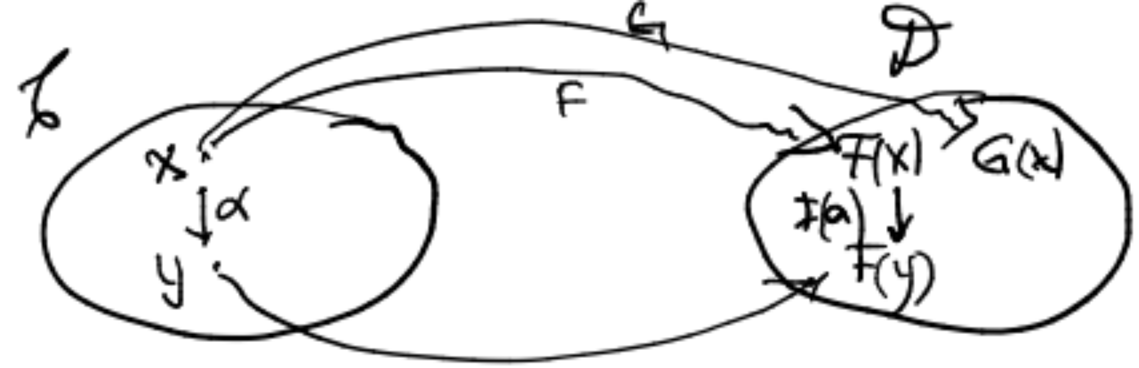
Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F , G : C -> D by functors.
A natural transformation $\mu : F \to G$ is the specification of, for each object $x \in Ob(\mathcal{C})$ a morphism $\mu_x : F(x) \to G(x)$ in $\mathcal{D}$ so that, $\forall$ morphisms $\alpha : x \to y$ in $\mathcal{C}$ the square

$$\begin{tikzcd} F(x) \arrow[r, "\mu_x"] \arrow[d, "F(\alpha)"'] & G(x) \arrow[d, "G(\alpha)"] \\ F(y) \arrow[r, "\mu_y"] & G(y) \end{tikzcd}$$

commutes.



Examples: Homomorphism of group representations and of quiver

Regarding repns of G as functors $\rho : G \to R\text{-mod}$ a homomorphism $\rho \to \sigma$ is a natural transformation $\rho \to \sigma$.

Regarding repns of a quiver Q as functors $F(Q)$, a homomorphism of repns $F_1 \to F_2$ is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

SCat has
- Objects : categories
- Morphisms : functors

Fun
- Objects : functors
- Morphisms : nat transfs

In fact SCat is a 2-category.

Example: Hom(C,D) where C and D are categories.

Let $\mathcal{C}$, $\mathcal{D}$ be categories. We define a category $\mathrm{Hom}_{\mathrm{SCat}}(\mathcal{C},\mathcal{D}) = \mathrm{Fun}(\mathcal{C},\mathcal{D})$

with objects $:=$ functors $\mathcal{C} \to \mathcal{D}$

morphisms $F \to G$ $:=$ natural transformations $F \to G$.

Implicit: we can compose natural transfs $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$

Then $(\lambda\mu)_x := \lambda_x \mu_x : F(x) \to H(x)$ in $\mathcal{D}$

There is a $1_F : F \to F$ for each functor F.

Pre-class Warm-up!!

Let M be an R-module. Is the operation that sends

$\mathrm{Hom}_R(R, M)$ to M

by sending

$$\phi \longmapsto \phi(1)$$

A a functor

B a natural transformation

C a category

D none of the above.

We have functors

$$\mathrm{Hom}_R(R,-) : R\text{-mod} \longrightarrow R\text{-mod}$$

$$1 : R\text{-mod} \longrightarrow R\text{-mod}.$$

$$\eta : \mathrm{Hom}(R,-) \longrightarrow 1$$

specified by

$$\eta_M : \mathrm{Hom}(R,M) \longrightarrow 1(M) = M$$

$$\eta_M(\phi) = \phi(1).$$

is a natural transformation

If $\alpha : M \to N$ then

$$\begin{tikzcd} \mathrm{Hom}(R,M) \arrow[r, "\eta_M"] \arrow[d, "\mathrm{Hom}_R(R,\alpha)"'] & M \arrow[d, "\alpha"] \\ \mathrm{Hom}(R,N) \arrow[r, "\eta_N"] & N \end{tikzcd}$$

commutes.

The double dual.

Let V be a finite dimensional vector space over a field k. Let V^* = Hom (V,k) be the vector space dual.

Question: Is the operation that sends V to V^*

A a functor?

B a natural transformation?

$$\begin{array}{ccc} k\text{-mod} & \longrightarrow & k\text{-mod} \\ V & \longmapsto & V^* \end{array}$$

is a contravariant functor i.e. a functor $k\text{-mod}^{op} \longrightarrow k\text{-mod}$

$F(V) = V^{**}$

$F : k\text{-mod} \to k\text{-mod}.$

$V \not\cong V^{**}$

We have natural transformation

$\eta : 1 \longrightarrow (-)^{**}$

$\eta_V : V \longrightarrow V^{**}$

$\eta_V(w) = (f \longmapsto f(w))$

for $w \in V$.

Also there is a nat. transfn $\Theta : (-)^{**} \longrightarrow V$ if V is finite dimensional. On the category where $\dim V < \infty \Rightarrow 1$ and $(-)^{**}$ are naturally isomorphic functors.

Question:
Is the functor V -> V^* naturally isomorphic to the identity functor on finite-dimensional vector spaces?

A Yes

B No ✓

1 is covariant : k-mod → k-mod

$(-)^*$ is contravariant.

Example.
Given a finite set X and a ring R we may construct the free R-module R^X on X. Let N be an R-module.
Consider the two functors Set -> Set specified by
if Y is a set then

F(Y) = $\mathrm{Hom}_{Set}(Y, N)$

G(Y) = $\mathrm{Hom}_R(R^Y, N)$.

Any map of sets $\phi : Y \to N$ extends uniquely to an R-module homomorphism

$\eta_Y(\phi) : R^Y \to N$. Here η_Y is a bijection of sets

$\eta_Y : F(Y) \to G(Y)$.

η is a natural isomorphism $F \cong G$

Pre-class Warm-up

Given a finite set X and a ring R we may construct the free R-module R^X on X.
Let f : M -> N be a homomorphism of R-modules.
Fixing X, consider the two functors R-mod -> Set specified by

$F(M) = \mathrm{Hom}_{Set}(X, M)$

$G(M) = \mathrm{Hom}_{R\text{-mod}}(R^X, M)$

On Wednesday we considered a bijection of sets $\theta_M : F(M) \to G(M)$.

Does the square commute?

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, leftrightarrow, "\theta_M"'] & F(N) \arrow[d, leftrightarrow, "\theta_N"] \\ G(M) \arrow[r, "G(f)"] & G(N) \end{tikzcd}$$

A Yes

If $\alpha : X \to M$ is a map of sets then
$F(f)(\alpha) := f\alpha$.
$G(f)$ is also postcomposition with f

Natural isomorphism, equivalence of categories.

If $F, G : \mathcal{C} \to \mathcal{D}$ are two functors a natural isomorphism is a natural transformation $\eta : F \to G$ for which $\exists$ a nat. transfn. $\theta : G \to F$ so that $\theta\eta = 1_F$, $\eta\theta = 1_G$.

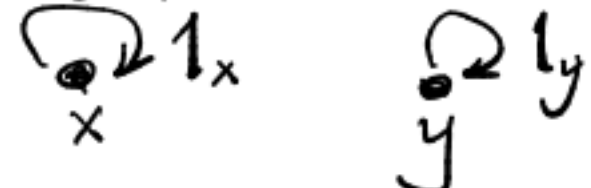
(Exercise: $\Leftrightarrow$ $\eta_x : F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$, $\forall x \in Ob(\mathcal{C})$.)

Definition: Categories $\mathcal{C}, \mathcal{D}$ are equivalent $\Leftrightarrow$ $\exists$ functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ plus natural isomorphisms $\theta : GF \xrightarrow{\simeq} 1_{\mathcal{C}}$ and $\psi : FG \xrightarrow{\simeq} 1_{\mathcal{D}}$.

If $\mathcal{C}, \mathcal{D}$ are isomorphic as categories they are equivalent

Example.
Let C be the category with two objects x and y, and with only the identity morphisms. Let k be a field.
We compare
the category of functors C -> k-mod
to the category of k x k -modules.

$\mathcal{C}$ is



A functor $\mathcal{C} \to k\text{-mod}$ is the specification of two k-vector spaces

Get a functor $F : Fun(\mathcal{C}, k\text{-mod}) \to k\times k\text{-mod}$

$k \times k = \{(u,v) \mid u, v \in k\}$

$F(A) =$ the $k\times k$-module $A(x) \oplus A(y)$

where $(u,v)\cdot(a,b) := (ua, vb)$

We define $G : k\times k\text{-mod} \to Fun(\mathcal{C}, k\text{-mod})$

$$G(B) = \begin{cases} x \longmapsto (1,0)\cdot B \\ y \longmapsto (0,1)\cdot B \end{cases}$$

$F(A) = A(x) \oplus A(y) \in k\times k\text{-mod}$

$G(B) = \begin{cases} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{cases}$

$\in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$.

$F : \mathrm{Fun}(\mathcal{C}, k) \rightleftarrows k\times k\text{-mod} : G$

$GF(A) = G(A(x) \oplus A(y))$

$= \begin{cases} x \longrightarrow (1,0)(A(x) \oplus A(y)) \\ \qquad\qquad \overset{\shortparallel}{(A(x) \oplus 0)} \\ y \longmapsto \quad 0 \oplus A(y). \end{cases}$

looks like A but is different.

$FG(B) = F\begin{pmatrix} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{pmatrix}$

$= (1,0)B \oplus (0,1)B$

$=$ set of pairs $((1,0)a, (0,1)b)$
$a, b \in B$.

looks like B but is different

F, G are not inverse isomorphisms

However $GF \cong 1_{\mathrm{Fun}(\mathcal{C}, k\text{-mod})}$

$FG \cong 1_{k\times k\text{-mod}}$

Define $\theta : 1_{\mathrm{Fun}(\ )} \longrightarrow GF$

$\theta_A : A \to GF(A)$ is morphism of functors $=$ a nat'l transfn.

$\theta_{A,x} : A(x) \to GF(A)(x) = (1,0)(A(x) \oplus A(y))$

$a \longmapsto (a, 0) \in A(x) \oplus A(y)$

Similarly we define $\psi : 1_{k\times k\text{-mod}} \to FG$.

$\mathrm{Fun}(G, k\text{-mod})$ and

$k \times k\text{-mod}$

might be isomorphic categories.

It's a job to show this (if true)

However the categories are equivalent. This is easier to show.

Speak of two categories being equivalent rather than isomorphic.

Reps of $\bullet \quad \bullet = Q$

'are the same as'

$k \times k$-modules.

$F(Q) = 1_x \circlearrowright \bullet_x \qquad \bullet_y \circlearrowleft 1_y$

Example.

Let $\mathcal{C} =$
$$\begin{tikzcd} x \arrow[loop right, "1_x"] \end{tikzcd}$$

Let $\mathcal{D} =$
$$\begin{tikzcd} x \arrow[loop left, "1_x"] \arrow[r, shift left, "\alpha"] & y \arrow[l, shift left, "\beta"] \arrow[loop right, "1_y"] \end{tikzcd}$$

where $\beta\alpha = 1_x \quad \alpha\beta = 1_y$

We show $\mathcal{C} \simeq \mathcal{D}$ are equivalent but $\mathcal{C} \not\cong \mathcal{D}$ (not isomorphic).

We have

$$\mathrm{inc} : \mathcal{C} \to \mathcal{D}$$
$$x \mapsto x$$
$$1_x \mapsto 1_x$$

and

$$F : \mathcal{D} \to \mathcal{C}$$
$$x \mapsto x$$
$$y \mapsto x$$
$$1_x, 1_y, \alpha, \beta \longmapsto 1_x .$$

$$F \circ \mathrm{inc} = 1_{\mathcal{C}} .$$

$$\mathrm{inc} \circ F \neq 1_{\mathcal{D}} .$$

Define $\theta : 1_{\mathcal{D}} \to \mathrm{inc} \circ F$

$$\theta_x : 1_{\mathcal{D}}(x) = x \mapsto \mathrm{inc} \circ F(x) = x$$

$$\theta_y : \quad y \longmapsto x$$

$$\theta_x := 1_x \quad \theta_y := \alpha .$$

Check this a nat. transformation.

It is iso (1_x, α are isomorphisms)

$\mathcal{C} \simeq \mathcal{D}$.

Pre-class Warm-up!!!

Let U, V : C -> D be functors. Suppose for each object x of C we have a map of sets (where z is some object of D)

$$f_x : \mathrm{Hom}_D(U(x), z) \longrightarrow \mathrm{Hom}_D(V(x), z)$$

Conditions A and B below both mean that f is natural with respect to x.
Which of A and B fits better with your understanding?

A For all morphisms g : x -> y in C the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_D(U(x), z) \arrow[d, "f_x"'] & \mathrm{Hom}_D(U(y), z) \arrow[l, "U(g)^*"'] \arrow[d, "f_y"] \\
\mathrm{Hom}_D(V(x), z) & \mathrm{Hom}_D(V(y), z) \arrow[l, "V(g)^*"']
\end{tikzcd}$$

B For all morphisms g : x -> y in C, whenever the next triangle commutes

$$\begin{tikzcd}
U(x) \arrow[rr, "U(g)"] \arrow[dr, "r"'] & & U(y) \arrow[dl, "s"] \\
& z &
\end{tikzcd}$$

the following triangle commutes.

$$\begin{tikzcd}
V(x) \arrow[rr, "V(g)"] \arrow[dr, "f_x(r)"'] & & V(y) \arrow[dl, "f_y(s)"] \\
& z &
\end{tikzcd}$$

Which do you prefer?

A Assume A. Show the second Δ in B commutes if the first does

B If the first commutes then $r = U(g)^*(s)$

so $f_x\, U(g)^*(s) = V(g)^* f_y(s)$

$f_x(r) = f_y(s) \cdot V(g)$. The second Δ commutes.

Adjoints

Definition.
Let F : C -> D and G : D -> C be functors.
We say F is the left adjoint of G and G is the right adjoint of F if there is a bijection (called the adjunction)

$$\mathrm{Hom}_{\mathcal{D}}(F(x), y) \to \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

natural in both x and y.

Examples. 1. $G : R\text{-mod} \to \text{Set}$

$M \longmapsto M$ as a set.

$F : \text{Set} \to R\text{-mod}$, $X \mapsto R^X$ = free module with X as basis.

$$\mathrm{Hom}_{R\text{-mod}}(R^X, M) \leftrightarrow \mathrm{Hom}_{\text{Set}}(X, M)$$

is a bijection natural in both X and M.

$X \longmapsto R^X$ is left adjoint to the forgetful functor $R\text{-mod} \to \text{Set}$.

2. Let R, S be rings and ${}_S M_R$ an (S,R)-bimodule.

Then $\mathrm{Hom}_S(M \otimes_R N, L)$

$$\cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, L))$$

where N is a left R-module, L is a left S-module, naturally in both N and L.

Here $F = M \otimes_R - : R\text{-mod} \to S\text{-mod}$

$G = \mathrm{Hom}_S(M, -) : S\text{-mod} \to R\text{-mod}$

and F is left adjoint to G.

Special case: R is a subring of S

$M = {}_S S_R$

Then $S \otimes_R N$: R-mod $\to$ S-mod

is left adjoint to

$\operatorname{Hom}_S(S_R, L)$: S-mod $\to$ R-mod

$\cong$

L as an R-mod.

$S \otimes_R -$ is left adjoint to

restriction : S-mod $\to$ R-mod.

The unit and counit of an adjunct ion.

Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then for $x \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) \longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(x, GF(x))$$

so

$$1_{F(x)} \longleftrightarrow \eta_x : x \to GF(x).$$

Write $\eta : 1_{\mathcal{C}} \to GF$

Also for $y \in \mathcal{D}$

$$\mathrm{Hom}_{\mathcal{C}}(G(y), G(y)) \longleftrightarrow \mathrm{Hom}_{\mathcal{D}}(FG(y), y)$$

$$1_{G(y)} \longleftrightarrow \epsilon_y : FG(y) \to y$$

Write $\epsilon : FG \to 1_{\mathcal{D}}$.

η is the unit of the adjunction

ϵ is the counit of the adjunction.

Examples.

1. $F(X) = R^X$ $G(M) = M$ as a set.

$GF(X) = R^X$ as a set.

$$\eta : 1_{set} \longrightarrow GF(X)$$

$\eta_X : X \longrightarrow R^X$ embeds X as the basis of R^X

$\epsilon : FG(M) = R^M \longrightarrow M$

is the homomorphism determined by sending basis element $m \in R^M$ to $m \in M$.

Lemma. The unit and counit are natural

The triangular identities

Proposition. The unit and counit determine the adjunction: natural transformations satisfying the triangular identities determine an adjunction.

Proposition. A left adjoint preserves epimorphisms. A right adjoint preserves monomorphisms.

Proposition. A left adjoint of a functor that preserves epimorphisms (and monos) sends projectives to projectives. Similarly, a right adjoint … injectives.