Category Theory
Eisenbud's Appendix 5 has the right topics
but is brief with a shortage of examples.

Definition. A category C is the specification of

1. A class of things called 'objects'. $x \in \text{Ob}(C)$ means x is an object of C.

2. For each pair $x, y \in \text{Ob}(C)$ we have a set $\text{Hom}_C(x,y)$ of things called morphisms.

3. A rule of composition

$$\text{Hom}(y,z) \times \text{Hom}(x,y) \longrightarrow \text{Hom}(x,z)$$
$$(g, f) \longmapsto gf \ (\text{or } g \circ f)$$

so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in \text{Ob}(C)$, there exists a morphism $1_x : x \to x$ so that $f 1_x = f \ \forall f : x \to y$, $1_x j = j \ \forall j : w \to x$

Morphism notation If $f \in \text{Hom}_C(x,y)$ we write $f : x \to y$ to denote this. x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples

1. Set $=$ category with objects $=$ sets, morphisms $=$ maps of sets.

Top $=$ category of topological-spaces, morphisms $=$ continuous maps

Group: morphisms $=$ group homoms

R-mod: Objects are R-modules, Morphisms are R-module homomorphisms.

2. A poset P may be regarded as a category P with $\text{Ob}(P) =$ elements of P, $\exists$ unique morphism $x \to y$ $\Rightarrow x \leq y$ in P.

Pre-class Warm-up!!!

Suppose f : M -> N is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms g, h : L -> M, if fg = fh then g = h.

B. For all pairs of homomorphisms g, h : N -> Q, if gf = hf then g = h.

C. Neither of the above.

B $\iff$ f is onto.

A. $L \overset{g}{\underset{h}{\rightrightarrows}} M \overset{f}{\to} N$ $\quad fg = fh$
$\Rightarrow g = h$

Proposition A $\iff$ f is 1-1.

Proof "A $\Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero.

Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A"   1-1 $\iff$ $\ker f = 0$

If $fg = fh$ then $\forall x \in L$, $fg(x) = fh(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\Leftrightarrow \exists$ a morphism $g: y \to x$

so that $gf = 1_x$ and $fg = 1_y$.

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may
construct a category $\mathcal{G}$
with only one object $*$
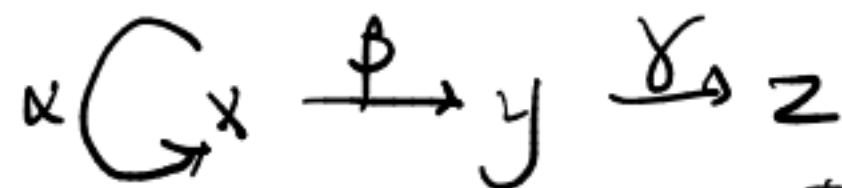and where $\text{Hom}(*,*) = G$
composition := multiplication
in G!

If M is a monoid we
construct a category $\mathcal{M}$
with one object $*$
$\text{Hom}(*,*) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)

$$\alpha \circlearrowright x \xrightarrow{\beta} y \xrightarrow{\gamma} z$$

Construct a category $F(Q)$ where
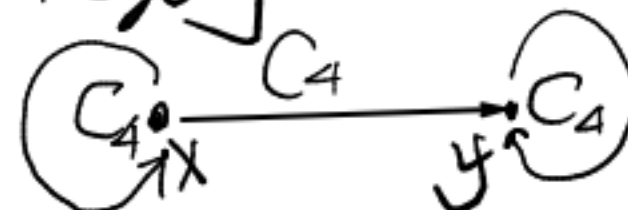$Ob(F(Q)) = $ vertices of Q

Morphisms = all possible words in the edges of Q where the end of a symbol = start of next.

Example $Ob = \{x, y, z\}$

Morphisms $= \{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \ldots\}$

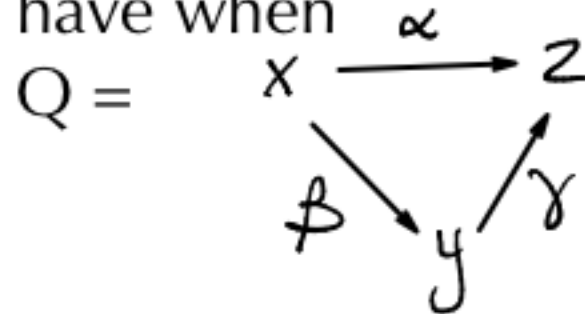
FI = the category with Objects = Finite sets,

Schematic description of a category with $Ob(\mathcal{C}) = \{x, y\}$

4 morphisms $x \to y$, $End(x) = C_4$
$End(y) = C_4$

$Hom_{\mathcal{C}}(x, x)$

Question:
How many morphisms does $F(Q)$ have when

$$Q = \quad x \xrightarrow{\alpha} z$$
$$\beta \searrow \quad \nearrow \gamma$$
$$y$$

A 3
B 4
C 5
D 6
E 7
F 8
G Infinitely many.

$C_4 \cdot \xrightarrow{\ C_4\ } \cdot C_4 \quad \longleftarrow \quad \{1_y, a_y \cdots\}$

$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$

$\{1, a, a^2, a^3\}$

Composition $\qquad a^2 \circ a_x = a^3$

$\qquad\qquad\qquad a_x^2 \circ a_x = a_x^3$

How many morphisms are there in the free category generated by the quiver
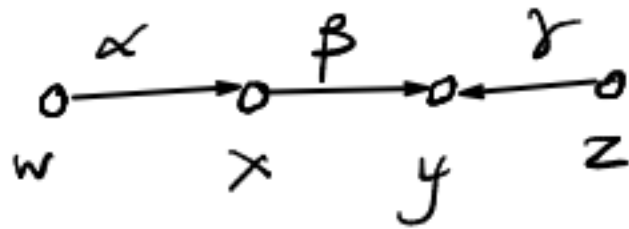
A 3

B 4

C 7

D 8

Constructions.

= The product of two categories $\mathcal{C}$, $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with objects (c,d)

$c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$

morphisms $(c,d) \xrightarrow{(f,g)} (c',d')$

where $f: c \to c'$ in $\mathcal{C}$, $g: d \to d'$ in $\mathcal{D}$.

= If $\mathcal{C}$ is a category, the opposite category is $\mathcal{C}^{op}$ with $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$

and morphisms $\bar{\alpha}$ where α is a morphism in $\mathcal{C}$. If $\alpha: x \to y$ then
$\bar{\alpha}: y \to x$. $\bar{\beta}: z \to y$, $\beta: y \to z$
$\bar{\alpha}\bar{\beta} := \overline{\beta\alpha}$

Question: Let I be the poset $0 \xrightarrow{\alpha} 1$

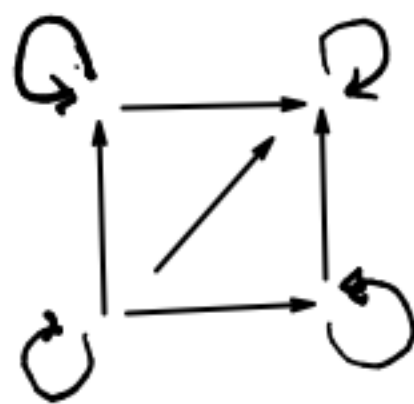How many morphisms does $I \times I$ have?

A 1
B 2
C 4
D 6
E 8
F 9 ✓

Functors

Definition. $\mathcal{C}, \mathcal{D}$ are categories.
the specification

of

- $\forall\, x \in Ob(\mathcal{C})$, an object $T(x) \in Ob(\mathcal{D})$

- $\forall$ morphisms $f : x \to y$ in $\mathcal{C}$,
  a morphism $T(f) : T(x) \to T(y)$ in $\mathcal{D}$

so that

1. $T(f\,g) = T(f)\,T(g)$

2. $T(1_x) = 1_{T(x)}$ $\quad \forall\, x \in Ob(\mathcal{C})$

Question: did we need to put in $T(1) = 1$
always, or did it follow from the other
axioms

Examples $F : AbGroups \to Groups$
Inclusion $F(A) = A$, $F(f) = f$

Forgetful functor like $F : Groups \to Set$
$F(G) = G$ regarded as a set
or $k[x]\text{-mod} \to k\text{-mod} =$ vector spaces over k.

If G and H are groups
we get categories $\mathcal{G}, \mathcal{H}$. A functor
$F : \mathcal{G} \to \mathcal{H}$ is 'the same thing as' a
group homomorphism $G \to H$.

If P and Q are posets a functor
$F : P \to Q$ is 'the same thing as'
an order preserving map $P \to Q$.

If X is a set let $R(X) =$ free R-module.
R is a commutative ring with X as a basis.
$R(-)$ is a functor $Set \to R\text{-mod}$

free group generated
by X
$F(-) : Set \to Group$ is a functor.

If M is a right R-module, L is a left R-module, we have functors

$$M \otimes - \;:\; R\text{-mod} \longrightarrow Ab\ Group$$

$$\operatorname{Hom}_R(L, -) \;:\; R\text{-mod} \longrightarrow Ab\ Group$$

R-mod = category of left R-modules

mod-R = right

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$
$$\beta \to \mathcal{D}$$

A contravariant functor has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor $\mathcal{C}^{op} \longrightarrow \mathcal{D}$.

If G is a group (or a monoid) a functor
F : G -> R-mod

A repn of G over R is a homomorphism $G \to GL(V)$ for some R-module V.

$F: G \to R\text{-mod}$ is
$$F(*) = V$$
$\forall\ g : * \to *,\ F(g) : V \to V$
is an R-module homomorphism.
$$F(g\,g^{-1}) = F(1_*) = 1_V : V \to V$$
$$= F(g)\,F(g^{-1})$$
$$F(g^{-1}) = F(g)^{-1}$$

Definition. A category C is small if Ob(C) is a set.

Example:

SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$$\rho: G \to GL(v) \qquad \sigma: G \to GL(w)$$

$$\rho \longrightarrow \sigma$$

Morphisms in the category Reps of G are R-linear maps $\theta: V \to W$

so that $\theta(\rho(g)(v)) = \sigma(g)(\theta(v))$.

i.e the square

commutes

$$\begin{CD} V @>{\rho(g)}>> V \\ @V{\theta}VV @VV{\theta}V \\ W @>{\sigma(g)}>> W \end{CD} \qquad \forall g \in G.$$

This defines a homomorphism of reps $\rho \to \sigma$

The functors between the category of representations of a group G and the category of RG-modules.

Reps of G are the same thing as RG-modules

RG = group ring of G
= free R-module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

$$\text{Reps of } G \rightleftarrows RG\text{-mod}$$

Given $\rho: G \to GL(v)$ get an RG-module V

$$\left(\sum_{g \in G} a_g g\right) \cdot v := \sum_{g \in G} a_g \, \rho(g)(v)$$

Let F : C -> D be a functor. Which of the following do you think means that F is an isomorphism of categories?

A For all objects x of C , x is isomorphic to F(x).

B For all pairs of objects x, y of C, F induces an isomorphism
Hom_C(x,y) ≈ Hom_D(F(x),F(y))

C There is a functor G : D -> C so that
FG = 1_D and   GF = 1_C.   ✓

D None of the above.

Example. The categorys.

Reps of G over R $\underset{G}{\overset{F}{\rightleftarrows}}$ RG-mod

are isomorphic.

Given $\rho : G \to GL(V)$ we get an RG-module $F(\rho) = V$, $\begin{cases} a_g g \cdot v \\ := \sum a_g \rho(g)(v) \end{cases}$

Given an RG-module W we get a homom. $G(W) : G \to GL(w)$
$g \longmapsto (w \longmapsto g \cdot v)$.

$GF(\rho) = \rho$
$FG(V) = F(G \to GL(v))$
$= V$ with original module action of RG.

If Q is a directed graph (= a quiver), a representation of Q over R is the specification of

- for each vertex x of Q, an R-module M_x
- for each arrow $x \xrightarrow{\alpha} y$ in Q an R-module homomorphism

$$M_x \xrightarrow{M(\alpha)} M_y$$

Example: $Q = \alpha \circlearrowleft \bullet x$, a repn M is an R-module M_x with an R-linear map $M(\alpha): M_x \to M_x$.

This is the same thing as an $R[t]$-module where t acts via α.

It is the same thing as a functor $F(Q) \to$ R-mod

A homomorphism of quiver representations the specification of, for each vertex x of $Q = x \in Ob \, F(Q)$ an R-module homomorphism

$$F_1(x) \xrightarrow{\mu_x} F_2(x)$$

$$\| \qquad\qquad \|$$

$$M_{1x} \qquad\qquad M_{2x}$$

so that every diagram commutes

$$
\begin{array}{ccc}
F_1(x) & \xrightarrow{\mu_x} & F_2(x) \\
\downarrow{F_1(\alpha)} & & \downarrow{F_2(\alpha)} \\
F_1(y) & \xrightarrow{\mu_y} & F_2(y)
\end{array}
$$

($\forall$ morphisms α in $F(Q)$)

Natural transformations

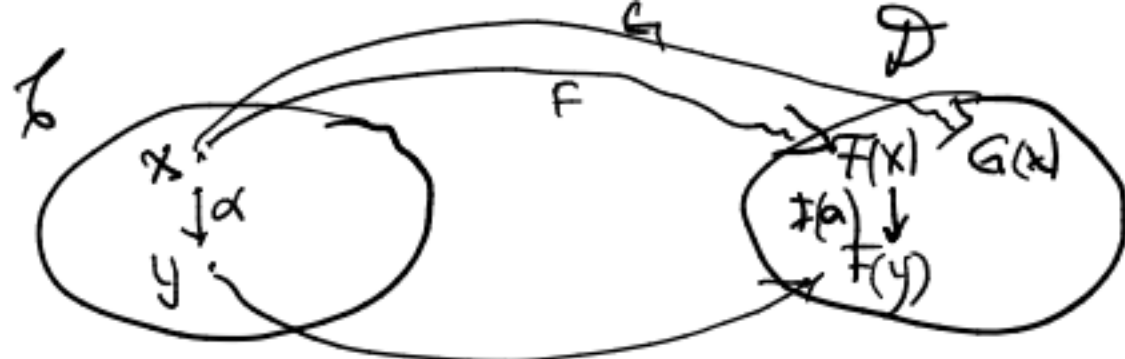These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F, $G : C \to D$ be functors. A natural transformation

$\mu : F \to G$ is the specification of, for each object $x \in \text{Ob}(C)$ a morphism $\mu_x : F(x) \to G(x)$ in D so that, $\forall$ morphisms $\alpha : x \to y$ in C the square

$$
\begin{array}{ccc}
F(x) & \xrightarrow{\mu_x} & G(x) \\
{\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\
F(y) & \xrightarrow{\mu_y} & G(y)
\end{array}
$$

commutes.

Examples: Homomorphism of group representations and of quiver

Regarding reps of G as functors $\rho : G \to R\text{-mod}$ a homomorphism $\rho \to \sigma$ is a natural transformation. $\rho \to \sigma$

Regarding reps of a quiver Q as functors $F(Q)$, a homomorphism of reps $F_1 \to F_2$ is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

SCat has
- Objects : categories
- Morphisms : functors

Fun
- Objects : functors
- Morphisms : nat transf

In fact SCat is a 2-category.

Example: $\text{Hom}(C,D)$ where C and D are categories.

Let $\mathcal{C}, \mathcal{D}$ be categories. We define a category $\text{Hom}_{SCat}(\mathcal{C}, \mathcal{D})$
$$= \text{Fun}(\mathcal{C}, \mathcal{D})$$
with objects : = functors $\mathcal{C} \to \mathcal{D}$

morphisms $F \to G$
:= natural transformation $F \to G$.

Implicit: we can compose natural transfos $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$ then $(\lambda \mu)_x := \lambda_x \mu_x : F(x) \to H(x)$ in $\mathcal{D}$

There is a $1_F : F \to F$ for each functor F.

Pre-class Warm-up!!

Let M be an R-module. Is the operation that sends

$$\text{Hom}_R(R, M) \quad \text{to} \quad M$$

by sending

$$\phi \longmapsto \phi(1)$$

A   a functor

B   a natural transformation

C   a category

D   none of the above.

We have functors

$$\text{Hom}_R(R, -) : R\text{-mod} \longrightarrow R\text{-mod}$$

$$1 \qquad : R\text{-mod} \longrightarrow R\text{-mod}.$$

$$\eta : \text{Hom}(R, -) \longrightarrow 1$$

specified by

$$\eta_M : \text{Hom}(R, M) \longrightarrow 1(M) = M$$

$$\eta_M(\phi) = \phi(1).$$

is a natural transformation.

If $\alpha : M \to N$ then
$$
\begin{array}{ccc}
\text{Hom}(R, M) & \xrightarrow{\eta_M} & M \\
\downarrow{\scriptstyle \text{Hom}_R(R, \alpha)} & & \downarrow{\scriptstyle \alpha} \\
\text{Hom}(R, N) & \xrightarrow{\eta_N} & N
\end{array}
$$

commutes.

The double dual.

Let V be a finite dimensional vector space over a field k. Let $V^* = \mathrm{Hom}(V,k)$ be the vector space dual.

Question: Is the operation that sends V to V^*

A a functor?

B a natural transformation?

$$k\text{-mod} \longrightarrow k\text{-mod}$$
$$V \longmapsto V^*$$

is a contravariant functor i.e. a functor $k\text{-mod}^{op} \longrightarrow k\text{-mod}$

$F(V) = V^{**}$

$F : k\text{-mod} \to k\text{-mod}$,

$V \not\cong V^{**}$

We have natural transformation

$\eta : 1 \longrightarrow (-)^{**}$

$\eta_V : V \longrightarrow V^{**}$

$\eta_V(w) = (f \longmapsto f(w))$ for $w \in V$.

Also there is a nat. transfn $\theta : (-)^{**} \longrightarrow V$ if V is finite dimensional. On the category where $\dim V < \infty \Rightarrow 1$ and $(-)^{**}$ are naturally isomorphic functors.

Question:
Is the functor $V \to V^*$ naturally isomorphic to the identity functor on finite-dimensional vector spaces?

A Yes

B No ✓

1 is covariant : $k\text{-mod} \to k\text{-mod}$

$(-)^*$ is contravariant.

Example.
Given a finite set X and a ring R we may construct the free R-module R^X on X. Let N be an R-module.

Consider the two functors Set -> Set specified by

If Y is a set then

$F(Y) = \text{Hom}_{Set}(Y, N)$

$G(Y) = \text{Hom}_R(R^Y, N)$.

Any map of sets $\phi : Y \to N$ extends uniquely to an R-module homomorphism

$\eta_Y(\phi) : R^Y \to N$. Here η_Y

is a bijection of sets

$\eta_Y : F(Y) \to G(Y)$.

η is a natural isomorphism $F \cong G$

Given a finite set X and a ring R we may construct the free R-module R^X on X.
Let $f : M \to N$ be a homomorphism of R-modules.
Fixing X, consider the two functors R-mod -> Set specified by

$F(M) = \mathrm{Hom}_{Set}(X , M)$

$G(M) = \mathrm{Hom}_{R\text{-mod}}(R^X, M)$

On Wednesday we considered a bijection of sets
$\theta_M : F(M) \to G(M)$.

Does the square commute?

$$
\begin{array}{ccc}
F(M) & \xrightarrow{F(f)} & F(N) \\
\theta_M \uparrow & & \theta_N \uparrow \\
G(M) & \xrightarrow{G(f)} & G(N)
\end{array}
$$

A   Yes

If $\alpha : X \to M$ is a map of sets then
$F(f)(\alpha) := f\alpha$.
$G(f)$ is also postcomposition with f

If $F, G : \mathcal{C} \to \mathcal{D}$ are two functors a natural isomorphism is a natural transformation $\eta : F \to G$ for which $\exists$ a nat. transfn. $\theta : G \to F$ so that $\theta\eta = 1_F$, $\eta\theta = 1_G$.

$\left(\begin{array}{l}\text{Exercise: } \Longleftrightarrow \eta_x : F(x) \to G(x) \text{ is}\\ \text{an isomorphism in } \mathcal{D}, \forall x \in Ob(\mathcal{C}).\end{array}\right)$

Definition: Categories $\mathcal{C}, \mathcal{D}$ are equivalent $\Longleftrightarrow \exists$ functors $F : \mathcal{C} \to \mathcal{D}$ $G : \mathcal{D} \to \mathcal{C}$ plus natural isomorphisms $\theta : GF \xrightarrow{\sim} 1_{\mathcal{C}}$ and $\psi : FG \xrightarrow{\sim} 1_{\mathcal{D}}$.

If $\mathcal{C}, \mathcal{D}$ are isomorphic as categories they are equivalent

Example.
Let C be the category with two objects x and y, and with only the identity morphisms. Let k be a field.
We compare
the category of functors C -> k-mod
to the category of k x k -modules.

$\mathcal{C}$: $\underset{x}{\overset{\curvearrowright}{\bullet}} 1_x \qquad \underset{y}{\overset{\curvearrowright}{\bullet}} 1_y$

A functor $\mathcal{C} \to$ k-mod is the specification of two k-vector spaces

Get a functor $F : Fun(\mathcal{C}, \text{k-mod}) \to k \times k\text{-mod}$ $k \times k = \{(u,v) \mid u, v \in k\}$
$F(A) = $ the $k \times k$-module $A(x) \oplus A(y)$
  where $(u,v) \cdot (a,b) := (ua, vb)$

We define $G : k \times k\text{-mod} \to Fun(\mathcal{C}, \text{k-mod})$

$G(B) = \begin{cases} x \longmapsto (1,0) \cdot B \\ y \longmapsto (0,1) \cdot B \end{cases}$

$$F(A) = A(x) \oplus A(y) \in k \times k\text{-mod}$$

$$G(B) = \begin{cases} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{cases} \in Fun(6, k\text{-mod}).$$

$$F: Fun(6, k) \leftrightarrows k \times k\text{-mod} : G$$

$$GF(A) = G\left(A(x) \oplus A(y)\right)$$

$$= \begin{cases} x \longmapsto (1,0)\left(A(x) \oplus A(y)\right) \\ \qquad \left(\ddot{A}(x) \oplus 0\right) \\ y \longmapsto 0 \oplus A(y). \end{cases}$$

looks like A but is different.

$$FG(B) = F\begin{pmatrix} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{pmatrix}$$

$$= (1,0)B \oplus (0,1)B.$$

$$= \text{set of pairs } ((1,0)a, (0,1)b)$$
$$a, b \in B.$$

looks like B but is different

F, G are not inverse isomorphisms.

However $GF \cong 1_{Fun(6, k\text{-mod})}$

$$FG \cong 1_{k \times k\text{-mod}}$$

Define $\Theta : 1_{Fun(\ )} \longrightarrow GF$

$\Theta_A : A \to GF(A)$ is morphism
of functors $=$ a nat'l transf^n.

$\Theta_{A,x} : A(x) \longrightarrow GF(A)(x) = (1,0)(A(x) \oplus A(y))$
$$a \longmapsto (a, 0) \in A(x) \oplus A(y)$$

Similarly we define $\psi : 1_{k \times k\text{-mod}} \to FG$.

$\text{Fun}(\mathcal{G}, k\text{-mod})$ and

$k \times k\text{-mod}$

might be isomorphic categories.

It's a job to show this (if true)

However the categories are
equivalent. This is easier to show.

Speak of two categories being
equivalent rather than isomorphic.

Repns of $\quad \bullet \quad \bullet \quad = Q$

`are the same as'

$k \times k$ -modules.

$F(Q) = 1_x \mathcal{G}_x \quad \bullet_y \mathcal{2}^1_y$

Example.

Let $C = $ x with 1_x

Let $D = $ $1_x \circlearrowleft x \underset{\beta}{\overset{\alpha}{\rightleftarrows}} y \circlearrowright 1_y$
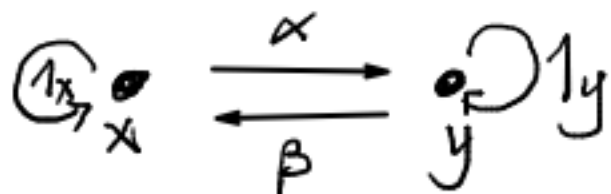
where $\beta\alpha = 1_x \qquad \alpha\beta = 1_y$

We show $C \simeq D$ are equivalent
but $C \not\cong D$ (not isomorphic).

We have

$inc: C \longrightarrow D$

$\qquad x \longmapsto x$

$\qquad 1_x \longmapsto 1_x$

and

$F: D \longrightarrow C$

$\qquad x \longmapsto x$

$\qquad y \longmapsto x$
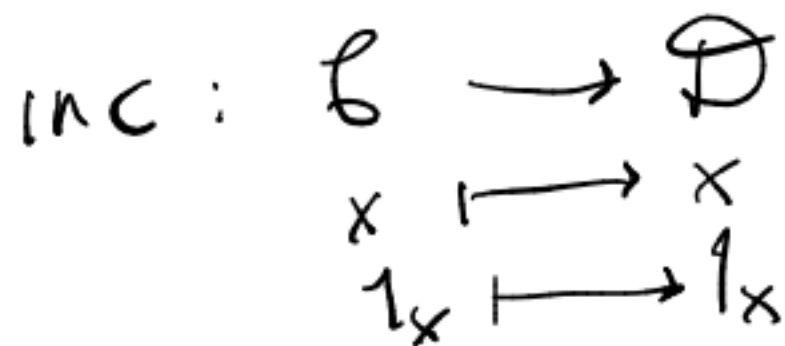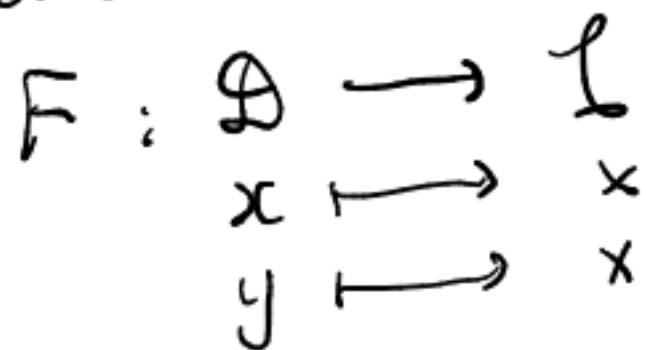
$1_x, 1_y, \alpha, \beta \longmapsto 1_x$

$F \circ inc = 1_C$

$inc \circ F \neq 1_D$

Define $\theta: 1_D \longrightarrow inc \circ F$

$\theta_x: 1_D(x) = x \longmapsto inc \circ F(x) = x$

$\theta_y: \qquad y \longmapsto x$

$\theta_x := 1_x \qquad \theta_y := \alpha$

Check this a nat'l. transformation

It is iso (1_x, α are isomorphisms)

$C \cong D$.

Pre-class Warm-up!!!

Let $U, V : C \to D$ be functors. Suppose for each object x of C we have a map of sets (where z is some object of D)

$$f_x : \mathrm{Hom}_D(U(x), z) \to \mathrm{Hom}_D(V(x), z)$$

Conditions A and B below both mean that f is natural with respect to x.

Which of A and B fits better with your understanding?

A For all morphisms $g : x \to y$ in C the following diagram commutes.

$$\mathrm{Hom}_D(U(x), z) \xleftarrow{U(g)^*} \mathrm{Hom}_D(U(y), z)$$
$$f_x \downarrow \qquad\qquad f_y \downarrow$$
$$\mathrm{Hom}_D(V(x), z) \xleftarrow{V(g)^*} \mathrm{Hom}_D(V(y), z)$$

B For all morphisms $g : x \to y$ in C, whenever the next triangle commutes

$$U(x) \xrightarrow{U(g)} U(y)$$
$$r \searrow \qquad \swarrow s$$
$$z$$

the following triangle commutes.

$$V(x) \xrightarrow{V(g)} V(y)$$
$$f_x(r) \searrow \qquad \swarrow f_y(s)$$
$$z$$

Which do you prefer?

A Assume A. Show the second $\triangle$ in B commutes if the first does

B If the first commutes then $r = U(g)^*(s)$

so $f_x \, U(g)^*(s) = V(g)^* f_y(s)$

$$f_x(r) = f_y(s) \cdot V(g). \text{ The second } \triangle \text{ commutes.}$$

Adjoints

Definition.
Let $F : C \to D$ and $G : D \to C$ be functors.
We say F is the left adjoint of G and G is
the right adjoint of F if there is a
bijection (called the adjunction)

$$\text{Hom}_{\mathcal{D}}(F(x), y) \longrightarrow \text{Hom}_{\mathcal{C}}(x, G(y))$$

natural in both x and y.

Examples. 1. $G : R\text{-mod} \to \text{Set}$
$$M \longmapsto M \text{ as a set.}$$

$F : \text{Set} \to R\text{-mod}, \quad X \longmapsto R^X = $
free module
with X as basis.

$$\text{Hom}_{R\text{-mod}}(R^X, M) \longleftrightarrow \text{Hom}_{\text{Set}}(X, M)$$

is a bijection natural in both
X and M.

$X \longmapsto R^X$ is left adjoint to the
forgetful functor $R\text{-mod} \to \text{Set}$.

2. Let R, S be rings and
$_S M_R$ an (S, R)-bimodule

Then $\text{Hom}_S(M \otimes_R N, L)$

$$\cong \text{Hom}_R(N, \text{Hom}_S(M, L))$$

where N is a left R-module
L is a left S-module; naturally
in both N and L.

Here $F = M \otimes_R - : R\text{-mod} \to S\text{-mod}$
$G = \text{Hom}_S(M, -) : S\text{-mod} \to R\text{-mod}$
and F is left adjoint to G.

Special case: R is a subring of S

$M = {}_S S_R$

Then $S \otimes_R N : R\text{-mod} \to S\text{-mod}$

is left adjoint to

$\text{Hom}_S(S_R, L) : S\text{-mod} \to R\text{-mod}$

$\phantom{\text{Hom}_S(} \| \text{IS}$

L as an R-mod.

$S \otimes_R -$ is left adjoint to

restriction: $S\text{-mod} \to R\text{-mod}$.

Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to
$G : \mathcal{D} \to \mathcal{C}$. Then for $x \in \mathcal{C}$,

$$\text{Hom}_{\mathcal{D}}(F(x), F(x)) \longleftrightarrow \text{Hom}_{\mathcal{C}}(x, GF(x))$$

$$\begin{array}{ccc} 1_{F(x)} & \longleftrightarrow & \eta_x : x \to GF(x) \end{array}$$

so

Write $\boxed{\eta : 1_{\mathcal{C}} \to GF}$

Also for $y \in \mathcal{D}$

$$\text{Hom}_{\mathcal{C}}(G(y), G(y)) \longleftrightarrow \text{Hom}_{\mathcal{D}}(FG(y), y)$$

$$\begin{array}{ccc} 1_{G(y)} & \longleftrightarrow & \epsilon_y FG(y) \to y \end{array}$$

Write $\boxed{\epsilon : FG \to 1_{\mathcal{C}}}$

η is the unit of the adjunction
ϵ is the counit of the adjunction.

Examples.

1. $F(x) = R^x$ $G(M) = M$ as a set.

$GF(x) = R^x$ as a set.

$$\eta : 1_{set} \longrightarrow GF(x)$$

$$\eta_x : X \longrightarrow R^X \quad \text{embeds } X$$
$$\text{as the basis of } R^X$$

$$\epsilon : FG(M) = R^M \longrightarrow M$$

is the homomorphism determined
by sending basis element $m \in R^M$
to $m \in M$.

Let U be a multiplicatively closed subset of a
commutative ring R and let M be an R-module.
Given that the functor
F(M) = M[U^{-1}]
is left adjoint to the inclusion functor
R[U^{-1}]-mod -> R-mod, do you think the map

$$\theta_M : M \longrightarrow M[U^{-1}]$$

$$M \longmapsto \frac{m}{1}$$

is

A the unit of the adjunction ✓

B the counit of the adjunction?

$$F: M \longmapsto M[U^{-1}]$$

$$R\text{-mod} \xrightarrow{} R[U^{-1}]\text{-mod}$$

$$\xleftarrow[\text{Forget} = G]{}$$

F is left adjoint to G.

Bijection.

$$\text{Hom}_{R[U^{-1}]}\left(M[U^{-1}], N\right)$$

$$\longleftrightarrow \text{Hom}_R(M, N).$$

Unit. $\eta : 1_{R\text{-mod}} \longrightarrow GF$

Counit $\varepsilon : FG \longrightarrow 1_{R[U^{-1}]\text{-mod}}.$

Lemma. The unit and counit are natural

Here $F: \mathscr{C} \to \mathscr{D}$ $G: \mathscr{D} \to \mathscr{C}$
and there is a bijection
$$f: \mathrm{Hom}_{\mathscr{D}}(F(x), y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(x, G(y))$$
natural in both x and y.

To get the unit take $y = F(x)$
$$f: \mathrm{Hom}(F(x), F(x)) \longrightarrow \mathrm{Hom}(x, GF(x))$$
$$\eta_x = f(1_{F(x)}) : x \longrightarrow GF(x).$$

To show that $\eta : 1_{\mathscr{C}} \longrightarrow GF$ is natural we verify $\forall \, \alpha: x \to y$ in $\mathscr{D}$ the diagram commutes:

$$
\begin{array}{ccc}
x & \xrightarrow{\alpha} & y \\
\eta_x \downarrow & \searrow^{\gamma} & \downarrow \eta_y \\
GF(x) & \xrightarrow{GF(\alpha)} & GF(y)
\end{array}
$$

Consider the diagram

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(\alpha)} & F(y) \\
1_{F(x)} \downarrow & \searrow^{\beta} & \downarrow 1_{F(y)} \\
F(x) & \xrightarrow{F(\alpha)} & F(y)
\end{array}
$$

The vertical arrows correspond to
η_x, η_y under f or f^{-1}

The diagram commutes!
We use naturality of f in each
variable to deduce: first diagram
commutes.
We get: $\eta_y \alpha = f(\beta)$ from the
top β in the second diagram.
b/c $f(1_{F(y)}) = \eta_y$
$$f(1_{F(y)} F(\alpha)) = \alpha^*(\eta_y) = \eta_y \circ \alpha .$$
Similarly: $f(F(\alpha) 1_{F(x)}) = GF(\alpha) \circ \eta_x$
B/c $1_{F(y)} F(\alpha) = F(\alpha) 1_{F(x)}$, these are equal $\square$

The unit and counit determine the adjunction.

Proposition. Let $F : C \to D$ be left adjoint to
$G : D \to C$, so that we have a bijection

$$f : \text{Hom}_C(Fx, y) \longrightarrow \text{Hom}_D(x, Gy), \quad \forall x, y$$

Let η and ϵ be the unit and counit of the
adjunction.
If $u : F(x) \to y$ in C then $f(u)$ is
the composite

$$x \xrightarrow{\eta x} GF(x) \xrightarrow{G(u)} G(y)$$

If $v : x \to G(y)$ in X then $f^{-1}(v)$ is
the composite

$$F(x) \xrightarrow{F(v)} FG(y) \xrightarrow{\epsilon y} y$$

Question: Do you think you could
prove this? (in the next 5 mins?)

Yes   No ✓

Proof

Write $u = u \circ 1_{F(x)}$

Now $f(u) = G(u)_* f(1_{F(x)})$
by Monday's pre-class warm up.
(naturality of f in 2nd variable).
This proves

Write $v = 1_{G(y)} \circ v$

to get $f^{-1}(v) = \overline{F(v)}^* f^{-1}(1_{G(y)})$.

The triangular identities

Proposition. Let $F : C \to D$ be left adjoint to $G : D \to C$.
Let η and ϵ be the unit and counit of the adjunction.
Then the following two triangles commute.

$$F \xrightarrow{1_F} F \qquad G \xrightarrow{1_G} G$$
$$F\eta \searrow \quad \nearrow \epsilon_F \qquad \eta_G \searrow \quad \nearrow G\epsilon$$
$$FGF \qquad\qquad GFG$$

Interpretation: The morphisms are natural transformations. Given $x \in C$ we have
$\eta_x : x \to GF(x)$ so $F\eta_x : F(x) \to FGF(x)$ is part of a nat. transf. $F\eta$.
Also $\epsilon_y : FG(y) \to y$. Take $F(x) = (y)$ to get $\epsilon_{F(x)} : FGF(x) \to F(x)$.
ϵ_F is this nat. transf.

Proof Given a morphism $\alpha : F(x) \to y$ in $\mathcal{D}$ the corresponding morphism in $\mathcal{C}$ is the composite
$$x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y) \qquad (*)$$

Given a morphism $\beta : x \to G(y)$ the corresponding morphism in $\mathcal{D}$ is
$$F(x) \xrightarrow{F(\beta)} FG(y) \xrightarrow{\epsilon_y} y$$

Take β to be $(*)$:
$$F(x) \xrightarrow{F(\eta_x)} FGF(x) \xrightarrow{FG(\alpha)} FG(y) \xrightarrow{\epsilon_y} y$$

Now take $y = F(x)$ and $\alpha = 1_{F(x)}$. Then
$FGF(x) \xrightarrow{FG(\alpha)} FG(y)$ is $FGF(x) \xrightarrow[1_{FGF(x)}]{FG 1_{F(x)}} FGF(x)$

We get $1_{F(x)} = F(x) \xrightarrow{F(\eta_x)} FGF(x) \xrightarrow{\epsilon_{F(x)}} F(x)$

Proposition. A left adjoint preserves epimorphisms. A right adjoint preserves monomorphisms.

Proposition. A left adjoint of a functor that preserves epimorphisms (and monos) sends projectives to projectives. Similarly, a right adjoint … injectives.