

## Category Theory

Eisenbud's Appendix 5 has the right topics but is brief with a shortage of examples.

Definition. A category  $\mathcal{C}$  is the specification of

1. A class of things called 'objects'  $x \in \text{Ob}(\mathcal{C})$  means  $x$  is an object of  $\mathcal{C}$ .
2. For each pair  $x, y \in \text{Ob}(\mathcal{C})$  we have a set  $\text{Hom}_{\mathcal{C}}(x, y)$  of things called morphisms.

3. A rule of composition

$$\text{Hom}(y, z) \times \text{Hom}(x, y) \longrightarrow \text{Hom}(x, z)$$

so that  $(g, f) \longmapsto gf$  (or  $g \circ f$ )

- a.  $(hg)f = h(gf)$  always, whenever it is defined.

- i. For all  $x \in \text{Ob}(\mathcal{C})$ , there exists a morphism  $1_x : x \rightarrow x$  so that  $f1_x = f \quad \forall f : x \rightarrow y, 1_x j = j \quad \forall j : w \rightarrow x$

Morphism notation If  $f \in \text{Hom}_{\mathcal{C}}(x, y)$  we write  $f : x \rightarrow y$  to denote this.  $x$  is the 'domain' of  $f$ ,  $y$  is the 'codomain' or 'target' of  $f$ .

Examples

1. Set = category with objects = sets  
morphisms = maps of sets.

Top = category of topological spaces  
morphisms = continuous maps

Group: morphisms = group homomorphisms

$R$ -mod: Objects are  $R$ -modules  
Morphisms are  $R$ -module homomorphisms

2. A poset  $P$  may be regarded as a category  $\mathcal{P}$  with  $\text{Ob}(\mathcal{P}) = \text{elements of } P$ ,  $\exists$  unique morphism  $x \rightarrow y \Rightarrow x \leq y$  in  $P$ .

# Pre-class Warm-up!!!

Suppose  $f : M \rightarrow N$  is a homomorphism of abelian groups. Which of the following conditions necessarily implies that  $f$  is one-to-one?

- A. For all pairs of homomorphisms  $g, h : L \rightarrow M$ , if  $fg = fh$  then  $g = h$ .
- B. For all pairs of homomorphisms  $g, h : N \rightarrow Q$ , if  $gf = hf$  then  $g = h$ .
- C. Neither of the above.

$B \Leftrightarrow f$  is onto.

$$A. \quad L \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} M \xrightarrow{f} N \quad fg = fh \\ \Rightarrow g = h$$

Proposition  $A \Leftrightarrow f$  is 1-1.

Proof "A  $\Rightarrow$  1-1" If  $f$  is not 1-1 then  $\ker f \neq 0$ . Take

$L = \ker f$ ,  $g : L \rightarrow M$  is inclusion,  $h : L \rightarrow M$  is zero

Then  $fg = fh = 0$  but  $g \neq h$  so A fails.

"1-1  $\Rightarrow$  A" 1-1  $\Leftrightarrow \ker f = 0$

If  $fg = fh$  then  $\forall x \in L$ ,  $fg(x) = fh(x)$ . Thus  $g(x) = h(x)$  b/c  $f$  is 1-1, so  $g = h$ .

Definition.

$\Leftrightarrow \exists$  a morphism  $g: y \rightarrow x$   
so that  $gf = 1_x$  and  $fg = 1_y$ .

The following is not equivalent to  $f$   
being an isomorphism.  
or  $\Leftrightarrow f$  is 1-1 and onto?  
 $\Downarrow$   
A, say  $f$  is a monomorphism  
 $\Uparrow$   
B, say  $f$  is an epimorphism

I just suggested  
it for the purposes  
of discussion.

More examples: a group, a monoid

Given a group  $G$  we may  
construct a category  $\mathcal{G}$   
with only one object  $*$   
and where  $\text{Hom}(*, *) = G$   
composition: = multiplication  
in  $G$ !

If  $M$  is a monoid we  
construct a category  $\mathcal{M}$   
with one object  $*$   
 $\text{Hom}(*, *) = M$ .

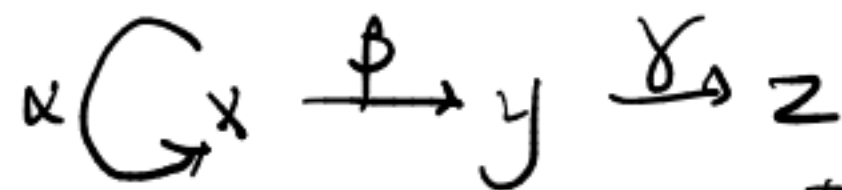
Question. Why do we take this definition of



More examples: weird categories.

Free categories

Let  $Q$  be a directed graph (quiver)



Construct a category  $F(Q)$  where  
 $Ob(F(Q)) = \text{vertices of } Q$

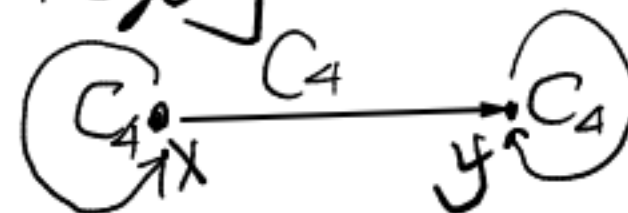
Morphisms = all possible words  
 in the edges of  $Q$  where  
 the end of a symbol = start of next.

Example  $Ob = \{x, y, z\}$

Morphisms =  $\{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \dots\}$

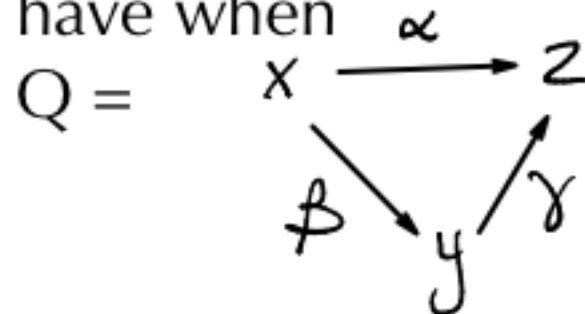
FI = the category with Objects = Finite sets,

Schematic description of  
 a category with  $Ob(e) = \{x, y\}$



4 morphisms  $x \rightarrow y$ ,  $End_x(x) = C_4$   
 $End_y(y) = C_4$   
 Question:  $Hom_x(x, x)$

How many morphisms does  $F(Q)$   
 have when



- A 3
- B 4
- C 5
- D 6
- E 7
- F 8
- G Infinitely many.



$$C_A = \{1_x, a_x, a_x^2, a_x^3\}$$

$$\{1, a, a^2, a^3\}$$

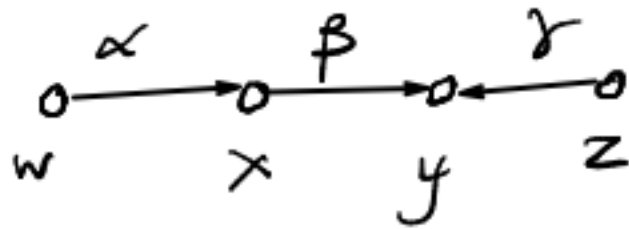
Composition

$$a^2 \circ a_x = a^3$$

$$a_x^2 \circ a_x = a_x^3$$

# Pre-class Warm-up!

How many morphisms are there in the free category generated by the quiver



A 3

B 4

C 7

D 8

## Constructions.

= The product of two categories  $\mathcal{C}, \mathcal{D}$  is the category

$\mathcal{C} \times \mathcal{D}$  with objects  $(c, d)$

$c \in \text{Ob}(\mathcal{C}), d \in \text{Ob}(\mathcal{D})$

morphisms  $(c, d) \xrightarrow{(f, g)} (c', d')$

where  $f: c \rightarrow c'$  in  $\mathcal{C}, g: d \rightarrow d'$  in  $\mathcal{D}$ .

= If  $\mathcal{C}$  is a category, the opposite category is  $\mathcal{C}^{\text{op}}$  with  $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$

and morphisms  $\bar{\alpha}$  where  $\alpha$  is a morphism in  $\mathcal{C}$ . If  $\alpha: x \rightarrow y$  then

$\bar{\alpha}: y \rightarrow x$ .  $\bar{\beta}: z \rightarrow y, \beta: y \rightarrow z$

$$\bar{\alpha} \bar{\beta} := \overline{\beta \alpha}$$

Question: Let  $I$  be the poset  $0 \xrightarrow{\alpha} 1$

How many morphisms does  $I \times I$  have?

A 1

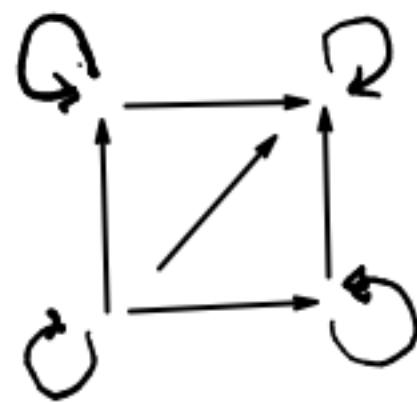
B 2

C 4

D 6

E 8

F 9 ✓



## Functors

Definition.  $\mathcal{C}, \mathcal{D}$  are categories.  
the specification

- $\forall x \in \text{Ob}(\mathcal{C})$ , an object  $T(x) \in \text{Ob}(\mathcal{D})$
- $\forall$  morphisms  $f: x \rightarrow y$  in  $\mathcal{C}$ ,  
a morphism  $T(f): T(x) \rightarrow T(y)$  in  $\mathcal{D}$   
so that

1.  $T(fg) = T(f)T(g)$   
2.  $T(1_x) = 1_{T(x)} \quad \forall x \in \text{Ob}(\mathcal{C})$

Question: did we need to put in  $T(1) = 1$  always, or did it follow from the other axioms

Examples  $F: \text{AbGroups} \rightarrow \text{Groups}$   
Inclusion  $F(A) = A, F(F) = F$

Forgetful functor like  $F: \text{Groups} \rightarrow \text{Set}$   
 $F(G) = G$  regarded as a set  
or  $R[x]\text{-mod} \rightarrow R\text{-mod} = \text{vector spaces over } R$ .

If  $G$  and  $H$  are groups  
we get categories  $\mathcal{G}, \mathcal{H}$ . A functor  
 $F: \mathcal{G} \rightarrow \mathcal{H}$  is 'the same thing as' a  
group homomorphism  $G \rightarrow H$ .

If  $P$  and  $Q$  are posets a functor  
 $F: P \rightarrow Q$  is 'the same thing as'  
an order preserving map  $P \rightarrow Q$ .

If  $X$  is a set let  $R(X) =$  free  $R$ -module.  
 $R$  is a commutative ring with  $X$  as a basis.  
 $R(-)$  is a functor  $\text{Set} \rightarrow R\text{-mod}$

$F(-): \text{Set} \rightarrow \text{Group}$  is a functor.  
free group generated by  $X$



If  $M$  is a right  $R$ -module,  $L$  is a left  $R$ -module, we have functors

$$M \otimes - : R\text{-mod} \rightarrow \text{Ab Group}$$

$$\text{Hom}_R(L, -) : R\text{-mod} \rightarrow \text{Ab Group}$$

$R\text{-mod}$  = category of left  $R$ -modules  
 $\text{mod-}R$  = right

These are covariant functors. The functor

$$\text{Hom}_R(-, L) : R\text{-mod} \rightarrow \text{Abelian groups}$$

$F$  is covariant means  $F(\alpha\beta) = F(\alpha)F(\beta)$

A contravariant functor  $L \rightarrow \mathcal{D}$  has the same definition except  $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor  $L^{\text{op}} \rightarrow \mathcal{D}$ .

If  $G$  is a group (or a monoid) a functor  $F : G \rightarrow R\text{-mod}$

A repr of  $G$  over  $R$  is a homomorphism  $G \rightarrow GL(V)$  for some  $R$ -module  $V$ .

$F : G \rightarrow R\text{-mod}$  is

$$F(*) = V$$

$\forall g : * \rightarrow *$ ,  $F(g) : V \rightarrow V$   
 is an  $R$ -module homomorphism.

$$F(gg^{-1}) = F(1_*) = 1_V : V \rightarrow V \\ = F(g)F(g^{-1})$$

$$F(g^{-1}) = F(g)^{-1}$$

Definition. A category  $C$  is small if  $\text{Ob}(C)$  is a set.

Example:

$\text{SCat}$  is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$$\rho: G \rightarrow \text{GL}(V) \quad \sigma: G \rightarrow \text{GL}(W)$$

$$\rho \rightarrow \sigma$$

Morphisms in the category  $\text{Reps of } G$

are  $R$ -linear maps  $\theta: V \rightarrow W$

$$\text{so that } \theta(\rho(g)(v)) = \sigma(g)(\theta(v))$$

i.e. the square

$$\begin{array}{ccc} V & \xrightarrow{\rho(g)} & V \\ \theta \downarrow & & \downarrow \theta \\ W & \xrightarrow{\sigma(g)} & W \end{array} \quad \text{commutes} \quad \forall g \in G.$$

This defines a homomorphism of reps  $\rho \rightarrow \sigma$

The functors between the category of representations of a group  $G$  and the category of  $RG$ -modules.

Reps of  $G$  are the same thing as  $RG$ -modules

$RG =$  group ring of  $G$   
 $=$  free  $R$ -module with elts of  $G$  as basis

Multn in  $RG$  is determined by group multiplication.

We get functors

$$\text{Reps of } G \quad \rightleftarrows \quad RG\text{-mod}$$

Given  $\rho: G \rightarrow \text{GL}(V)$

get an  $RG$ -module  $V$   
 $(\sum_{g \in G} a_g g) \cdot v := \sum_{g \in G} a_g \rho(g)(v)$

# Pre-class Warm-up!

Let  $F : C \rightarrow D$  be a functor. Which of the following do you think means that  $F$  is an isomorphism of categories?

- A For all objects  $x$  of  $C$ ,  $x$  is isomorphic to  $F(x)$ .
- B For all pairs of objects  $x, y$  of  $C$ ,  $F$  induces an isomorphism  $\text{Hom}_C(x, y) \approx \text{Hom}_D(F(x), F(y))$
- C There is a functor  $G : D \rightarrow C$  so that  $FG = 1_D$  and  $GF = 1_C$ . ✓
- D None of the above.

Example. The categories

Reps of  $G$  over  $R$   $\begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array}$   $RG\text{-mod}$

are isomorphic.

Given  $\rho : G \rightarrow GL(V)$  we get an  $RG$ -module  $F(\rho) = V := \sum a_g g \cdot v := \sum a_g \rho(g)(v)$

Given an  $RG$ -module  $W$  we get a homom.  $G(W) : G \rightarrow GL(W)$   
 $g \mapsto (w \mapsto g \cdot w)$ .

$$GF(\rho) = \rho$$

$$FG(V) = F(G \rightarrow GL(V)) = V \text{ with original module action of } RG.$$

If  $Q$  is a directed graph (= a quiver), a representation of  $Q$  over  $R$  is the specification of

- for each vertex  $x$  of  $Q$ , an  $R$ -module  $M_x$
- for each arrow  $x \xrightarrow{\alpha} y$  in  $Q$  an  $R$ -module homomorphism  $M_x \xrightarrow{M(\alpha)} M_y$

Example:  $Q = \alpha \circlearrowleft x$  a repr  $M$  is an  $R$ -module  $M_x$  with an  $R$ -linear map  $M(\alpha): M_x \rightarrow M_x$ .  
 This is the same thing as an  $R[t]$ -module where  $t$  acts via  $\alpha$ .

It is the same thing as a functor  $F(Q) \rightarrow R\text{-mod}$   
 A homomorphism of quiver representations

of, for each vertex  $x$  of  $Q = x \in \text{Ob } F(Q)$  the specification an  $R$ -module homomorphism

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\mu_x} & F_2(x) \\ \parallel & & \parallel \\ M_{1,x} & & M_{2,x} \end{array}$$

so that every diagram commutes

$$\begin{array}{ccc} F_1(x) & \xrightarrow{\mu_x} & F_2(x) \\ F_1(\alpha) \downarrow & & \downarrow F_2(\alpha) \\ F_1(y) & \xrightarrow{\mu_y} & F_2(y) \end{array}$$

( $\forall$  morphisms  $\alpha$  in  $F(Q)$ )



## Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

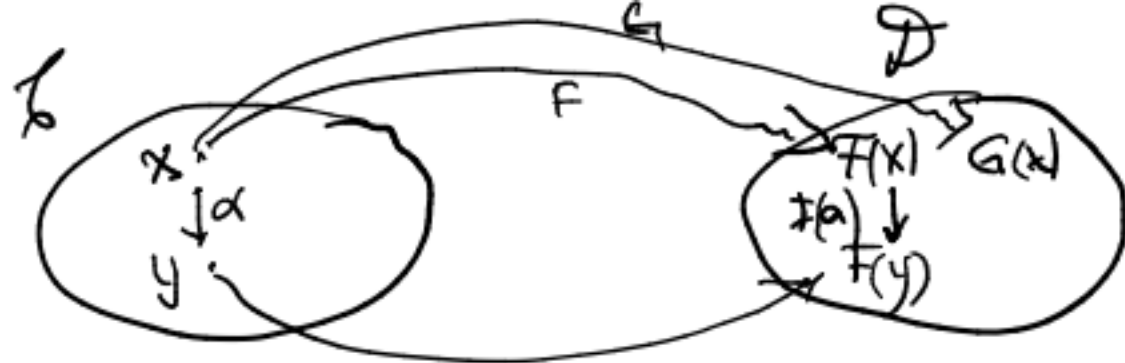
Definition. Let  $F, G : C \rightarrow D$  be functors.

A natural transformation

$\mu : F \rightarrow G$  is the specification of, for each object  $x \in \text{Ob}(C)$  a morphism  $\mu_x : F(x) \rightarrow G(x)$  in  $D$  so that,  $\forall$  morphisms  $\alpha : x \rightarrow y$  in  $C$  the square

$$\begin{array}{ccc} F(x) & \xrightarrow{\mu_x} & G(x) \\ F(\alpha) \downarrow & & G(\alpha) \downarrow \\ F(y) & \xrightarrow{\mu_y} & G(y) \end{array}$$

commutes.



Examples: Homomorphism of group representations and of quiver

Regarding reps of  $G$  as functors  $\rho : G \rightarrow R\text{-mod}$  a homomorphism  $\rho \rightarrow \sigma$  is a natural transformation.

Regarding reps of a quiver  $Q$  as functors  $F(Q)$ , a homomorphism of reps  $F_1 \rightarrow F_2$  is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

$\text{SCat}$  has

- Objects: categories
- Morphisms: functors

$\text{Fun}$

- Objects: functors
- Morphisms: nat transfs

In fact  $\text{SCat}$  is a 2-category.

Example:  $\text{Hom}(C, D)$  where  $C$  and  $D$  are categories.

Let  $\mathcal{C}, \mathcal{D}$  be categories. We define a category  $\text{Hom}_{\text{SCat}}(\mathcal{C}, \mathcal{D}) = \text{Fun}(\mathcal{C}, \mathcal{D})$  with objects: = functors  $\mathcal{C} \rightarrow \mathcal{D}$

morphisms  $F \rightarrow G$   
:= natural transformations  $F \rightarrow G$ .

Implicit: we can compose natural transfs  $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$   
then  $(\lambda\mu)_x := \lambda_x \mu_x : F(x) \rightarrow H(x)$   
in  $\mathcal{D}$

There is a  $1_F : F \rightarrow F$  for each functor  $F$ .

# Pre-class Warm-up!!

Let  $M$  be an  $R$ -module. Is the operation that sends

$\text{Hom}_R(R, M)$  to  $M$

by sending

$$\phi \longmapsto \phi(1)$$

- A a functor
- B a natural transformation
- C a category
- D none of the above.

We have functors  
 $\text{Hom}_R(R, -): R\text{-mod} \rightarrow R\text{-mod}$   
 $1: R\text{-mod} \rightarrow R\text{-mod}.$

$$\eta: \text{Hom}(R, -) \rightarrow 1$$

specified by

$$\eta_M: \text{Hom}(R, M) \rightarrow 1(M) = M$$

$$\eta_M(\phi) = \phi(1).$$

is a natural transformation.

If  $\alpha: M \rightarrow N$  then

$$\begin{array}{ccc} \text{Hom}(R, M) & \xrightarrow{\eta_M} & M \\ \downarrow \text{Hom}(R, \alpha) & & \downarrow \alpha \\ \text{Hom}(R, N) & \xrightarrow{\eta_N} & N \end{array}$$

commutes.

## The double dual.

Let  $V$  be a finite dimensional vector space over a field  $k$ . Let  $V^{\wedge*} = \text{Hom}(V, k)$  be the vector space dual.

Question: Is the operation that sends  $V$  to  $V^{\wedge*}$

A a functor?

B a natural transformation?

$$\begin{array}{ccc} k\text{-mod} & \longrightarrow & k\text{-mod} \\ V & \longmapsto & V^* \end{array}$$

is a contravariant functor i.e.  
a functor  $k\text{-mod}^{\text{op}} \rightarrow k\text{-mod}$

$$F(V) = V^{**}$$

$$F: k\text{-mod} \rightarrow k\text{-mod},$$

$$V \neq V^{**}$$

We have natural transformation

$$\eta: 1 \rightarrow (-)^{**}$$

$$\eta_V: V \rightarrow V^{**}$$

$$\eta_V(w) = (f \mapsto f(w))$$

for  $w \in V$ .

Also there is a nat. transfn  
 $\theta: (-)^{**} \rightarrow 1$  if  $V$  is  
finite dimensional. On the category where  
 $\dim V < \infty \Rightarrow 1$  and  $(-)^{**}$  are  
naturally isomorphic functors.



Question:

Is the functor  $V \rightarrow V^*$  naturally isomorphic to the identity functor on finite-dimensional vector spaces?

A Yes

B No ✓

$\uparrow$  is covariant:  $k\text{-mod} \rightarrow k\text{-mod}$   
 $(-)^*$  is contravariant.

Example.

Given a finite set  $X$  and a ring  $R$  we may construct the free  $R$ -module  $R^X$  on  $X$ . Let  $N$  be an  $R$ -module.

Consider the two functors  $\text{Set} \rightarrow \text{Set}$  specified by if  $Y$  is a set then

$$F(Y) = \text{Hom}_{\text{Set}}(Y, N)$$

$$G(Y) = \text{Hom}_R(R^Y, N)$$

Any map of sets  $\phi: Y \rightarrow N$  extends uniquely to an  $R$ -module homomorphism

$$\eta_Y(\phi): R^Y \rightarrow N \quad \text{Here } \eta_Y$$

is a bijection of sets

$$\eta_Y: F(Y) \rightarrow G(Y)$$

$\eta$  is a natural isomorphism  $F \cong G$

# Pre-class Warm-up

Given a finite set  $X$  and a ring  $R$  we may construct the free  $R$ -module  $R^X$  on  $X$ .

Let  $f: M \rightarrow N$  be a homomorphism of  $R$ -modules.

Fixing  $X$ , consider the two functors  $R\text{-mod} \rightarrow \text{Set}$  specified by

$$F(M) = \text{Hom}_{\text{Set}}(X, M)$$

$$G(M) = \text{Hom}_{R\text{-mod}}(R^X, M)$$

On Wednesday we considered a bijection of sets  $\theta_M: F(M) \rightarrow G(M)$ .

Does the square commute?

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ \theta_M \uparrow & & \theta_N \uparrow \\ G(M) & \xrightarrow{G(f)} & G(N) \end{array}$$

A Yes

If  $\alpha: X \rightarrow M$  is a map of sets then  
 $F(f)(\alpha) = f\alpha$ .  
 $G(f)$  is also postcomposition with  $f$

## Natural isomorphism, equivalence of categories.

If  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  are two functors a natural isomorphism is a natural transformation

$\eta: F \rightarrow G$  for which  $\exists$  a nat. transfn.  $\theta: G \rightarrow F$  so that  $\theta\eta = 1_F$ ,  $\eta\theta = 1_G$ .

(Exercise:  $\Leftrightarrow \eta_x: F(x) \rightarrow G(x)$  is an isomorphism in  $\mathcal{D}$ ,  $\forall x \in \text{Ob}(\mathcal{C})$ .)

Definition: Categories  $\mathcal{C}, \mathcal{D}$  are equivalent  $\Leftrightarrow \exists$  functors  $F: \mathcal{C} \rightarrow \mathcal{D}$

$G: \mathcal{D} \rightarrow \mathcal{C}$  plus natural isomorphisms

$\theta: GF \xrightarrow{\sim} 1_{\mathcal{C}}$  and  $\psi: FG \xrightarrow{\sim} 1_{\mathcal{D}}$ .

If  $\mathcal{C}, \mathcal{D}$  are isomorphic as categories they are equivalent

Example.

Let  $\mathcal{C}$  be the category with two objects  $x$  and  $y$ , and with only the identity morphisms. Let  $k$  be a field.

We compare

the category of functors  $\mathcal{C} \rightarrow k\text{-mod}$  to the category of  $k \times k$ -modules.

$\mathcal{C}$  is  $\begin{matrix} \circ & \xrightarrow{1_x} & \circ \\ x & & y \end{matrix}$

A functor  $\mathcal{C} \rightarrow k\text{-mod}$  is the specification of two  $k$ -vector spaces. Get a functor  $F: \text{Fun}(\mathcal{C}, k\text{-mod}) \rightarrow k \times k\text{-mod}$   
 $k \times k = \{(u, v) \mid u, v \in k\}$

$F(A) =$  the  $k \times k$ -module  $A(x) \oplus A(y)$

where  $(u, v) \cdot (a, b) := (ua, vb)$

We define  $G: k \times k\text{-mod} \rightarrow \text{Fun}(\mathcal{C}, k\text{-mod})$

$G(B) = \begin{cases} x \longmapsto (1, 0) \cdot B \\ y \longmapsto (0, 1) \cdot B \end{cases}$

$$F(A) = A(x) \oplus A(y) \in R \times R\text{-mod}$$

$$G(B) = \begin{cases} x \mapsto (1,0)B \\ y \mapsto (0,1)B \end{cases} \\ \in \text{Fun}(B, R\text{-mod}).$$

$$F: \text{Fun}(B, R) \xrightarrow{\cong} R \times R\text{-mod}: G$$

$$GF(A) = G(A(x) \oplus A(y)) \\ = \begin{cases} x \mapsto (1,0)(A(x) \oplus A(y)) \\ \quad \quad \quad (A(x) \oplus 0) \\ y \mapsto 0 \oplus A(y). \end{cases}$$

looks like A but is different.

$$FG(B) \cong F \left( \begin{array}{l} x \mapsto (1,0)B \\ y \mapsto (0,1)B \end{array} \right)$$

$$= (1,0)B \oplus (0,1)B.$$

$$= \text{set of pairs } ((1,0)a, (0,1)b) \\ a, b \in B.$$

looks like B but is different

F, G are not inverse isomorphisms

$$\text{However } GF \cong 1_{\text{Fun}(B, R\text{-mod})}$$

$$FG \cong 1_{R \times R\text{-mod}}$$

$$\text{Define } \theta: 1_{\text{Fun}(B, R\text{-mod})} \rightarrow GF$$

$\theta_A: A \rightarrow GF(A)$  is morphism of functors  $\cong$  a nat'l transfn.

$$\theta_{Ax}: A(x) \rightarrow GF(A)(x) = (1,0)(A(x) \oplus A(y)) \\ a \mapsto (a, 0) \in A(x) \oplus A(y)$$

Similarly we define  $\psi: 1_{R \times R\text{-mod}} \rightarrow FG$ .



$\text{Fun}(G, k\text{-mod})$  and  
 $k \times k\text{-mod}$

might be isomorphic categories.

It's a job to show this (if true)

However the categories are  
equivalent. This is easier to show.

Speak of two categories being  
equivalent rather than isomorphic.

Reprs of  $\bullet \bullet \bullet = Q$

are the same as

$k \times k$ -modules.

$$F(Q) = 1 \times \begin{matrix} \bullet \\ \circlearrowleft \\ \bullet \\ \circlearrowright \\ \bullet \end{matrix}_x$$

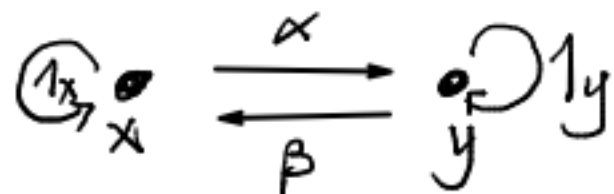
$$\begin{matrix} \bullet \\ \circlearrowleft \\ \bullet \\ \circlearrowright \\ \bullet \end{matrix} \begin{matrix} \circlearrowright \\ \circlearrowleft \\ \circlearrowright \\ \circlearrowleft \\ \circlearrowright \end{matrix} 1_y$$

Example.

Let  $C =$



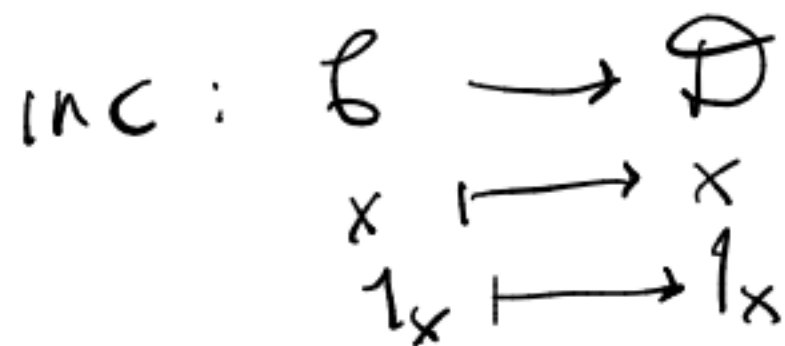
Let  $D =$



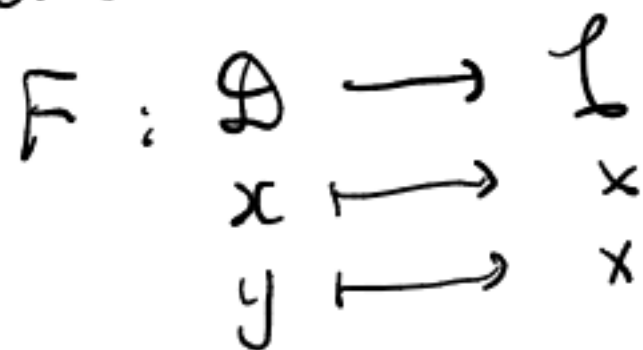
where  $\beta\alpha = 1_x$   $\alpha\beta = 1_y$

We show  $\mathcal{C} \simeq \mathcal{D}$  are equivalent  
but  $\mathcal{C} \not\cong \mathcal{D}$  (not isomorphic).

We have



and



$$1_x, 1_y, \alpha, \beta \longmapsto 1_x$$

$$F \circ \text{inc} = 1_{\mathcal{C}}$$

$$\text{inc} \circ F \neq 1_{\mathcal{D}}$$

Define  $\theta : 1_{\mathcal{D}} \longrightarrow \text{inc} \circ F$

$$\theta_x : 1_{\mathcal{D}}(x) = x \longmapsto \text{inc} \circ F(x) = x$$

$$\theta_y : y \longmapsto x$$

$$\theta_x := 1_x \quad \theta_y := \alpha$$

Check this a nat. transformation

It is iso ( $1_x, \alpha$  are isomorphisms)

$$\mathcal{C} \simeq \mathcal{D}$$

# Pre-class Warm-up!!!

Let  $U, V : C \rightarrow D$  be functors. Suppose for each object  $x$  of  $C$  we have a map of sets (where  $z$  is some object of  $D$ )

$$f_x : \text{Hom}_D(U(x), z) \rightarrow \text{Hom}_D(V(x), z)$$

Conditions A and B below both mean that  $f$  is natural with respect to  $x$ .

Which of A and B fits better with your understanding?

A For all morphisms  $g : x \rightarrow y$  in  $C$  the following diagram commutes.

$$\begin{array}{ccc} \text{Hom}_D(U(x), z) & \xrightarrow{U(g)^*} & \text{Hom}_D(U(y), z) \\ f_x \downarrow & & f_y \downarrow \\ \text{Hom}_D(V(x), z) & \xrightarrow{V(g)^*} & \text{Hom}_D(V(y), z) \end{array}$$

B For all morphisms  $g : x \rightarrow y$  in  $C$ , whenever the next triangle commutes

$$\begin{array}{ccc} U(x) & \xrightarrow{U(g)} & U(y) \\ r \searrow & & \swarrow s \\ & z & \end{array}$$

the following triangle commutes.

$$\begin{array}{ccc} V(x) & \xrightarrow{V(g)} & V(y) \\ f_x(r) \searrow & & \swarrow f_y(s) \\ & z & \end{array}$$

Which do you prefer?

A Assume A. Show the second  $\Delta$  in B commutes if the first does  
 B If the first commutes then  $r = U(g)^*(s)$   
 so  $f_x U(g)^*(s) = V(g)^* f_y(s)$   
 $f_x(r) = f_y(s) \cdot V(g)$ . The second  $\Delta$  commutes.

## Adjoint

Definition.

Let  $F: C \rightarrow D$  and  $G: D \rightarrow C$  be functors.

We say  $F$  is the left adjoint of  $G$  and  $G$  is

the right adjoint of  $F$  if there is a

bijection (called the adjunction)

$$\text{Hom}_D(F(x), y) \rightarrow \text{Hom}_C(x, G(y))$$

natural in both  $x$  and  $y$ .

Examples. 1.  $G: R\text{-mod} \rightarrow \text{Set}$

$$M \mapsto M \text{ as a set.}$$

$F: \text{Set} \rightarrow R\text{-mod}$ ,  $X \mapsto R^X =$   
free module  
with  $X$  as basis.

$$\text{Hom}_{R\text{-mod}}(R^X, M) \leftrightarrow \text{Hom}_{\text{Set}}(X, M)$$

$\delta$  a bijection natural in both  
 $X$  and  $M$ .

$X \mapsto R^X$  is left adjoint to the  
forgetful functor  $R\text{-mod} \rightarrow \text{Set}$ .

2. Let  $R, S$  be rings and  
 ${}_S M_R$  an  $(S, R)$ -bimodule

Then  $\text{Hom}_S(M \otimes_R N, L)$

$$\cong \text{Hom}_R(N, \text{Hom}_S(M, L))$$

where  $N$  is a left  $R$ -module  
 $L$  is a left  $S$ -module, naturally  
in both  $N$  and  $L$ .

Here  $F = M \otimes_R -: R\text{-mod} \rightarrow S\text{-mod}$

$G = \text{Hom}_S(M, -): S\text{-mod} \rightarrow R\text{-mod}$

and  $F$  is left adjoint to  $G$ .



Special case:  $R$  is a subring of  $S$

$$M = {}_S S_R$$

Then  $S \otimes_R N : R\text{-mod} \rightarrow S\text{-mod}$

is left adjoint to

$$\text{Hom}_S(S_R, L) : S\text{-mod} \rightarrow R\text{-mod}$$

is

$L$  as an  $R$ -mod.

$S \otimes_R -$  is left adjoint to

restriction:  $S\text{-mod} \rightarrow R\text{-mod}$ .

The unit and counit of an adjunction.

Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be left adjoint to  $G: \mathcal{D} \rightarrow \mathcal{C}$ . Then for  $x \in \mathcal{C}$ ,

$$\text{Hom}_{\mathcal{D}}(F(x), F(x)) \leftrightarrow \text{Hom}_{\mathcal{C}}(x, GF(x))$$

so  $\text{id}_{F(x)} \leftrightarrow \eta_x: x \rightarrow GF(x)$ .

Write  $\eta: \text{id}_{\mathcal{C}} \rightarrow GF$

Also for  $y \in \mathcal{D}$

$$\text{Hom}_{\mathcal{C}}(G(y), G(y)) \leftrightarrow \text{Hom}_{\mathcal{D}}(FG(y), y)$$

$\text{id}_{G(y)} \leftrightarrow \epsilon_y: FG(y) \rightarrow y$

Write  $\epsilon: FG \rightarrow \text{id}_{\mathcal{D}}$

$\eta$  is the unit of the adjunction

$\epsilon$  is the counit of the adjunction.

Examples.

1.  $F(x) = \mathbb{R}^x$      $G(M) = M$  as a set.

$GF(x) = \mathbb{R}^x$  as a set.

$$\eta: \text{id}_{\text{Set}} \rightarrow GF(x)$$

$$\eta_x: x \rightarrow \mathbb{R}^x \text{ embeds } x \text{ as the basis of } \mathbb{R}^x$$

$$\epsilon: FG(M) = \mathbb{R}^M \rightarrow M$$

is the homomorphism determined by sending basis element  $m \in \mathbb{R}^M$  to  $m \in M$ .

# Pre-class Warm-up!!

Let  $U$  be a multiplicatively closed subset of a commutative ring  $R$  and let  $M$  be an  $R$ -module.

Given that the functor

$$F(M) = M[U^{-1}]$$

is left adjoint to the inclusion functor

$R[U^{-1}]\text{-mod} \rightarrow R\text{-mod}$ , do you think the map

$$\theta_M: M \longrightarrow M[U^{-1}]$$
$$M \longleftarrow \frac{M}{1}$$

is

A the unit of the adjunction ✓

B the counit of the adjunction?

$$F: M \longleftarrow M[U^{-1}]$$
$$R\text{-mod} \longrightarrow R[U^{-1}]\text{-mod}$$
$$\longleftarrow \text{Forget} = G$$

$F$  is left adjoint to  $G$ .

Bijection

$$\text{Hom}_{R[U^{-1}]}(M[U^{-1}], N) \longleftrightarrow \text{Hom}_R(M, N)$$

Unit  $\eta: 1_{R\text{-mod}} \longrightarrow GF$

Counit  $e: FG \longrightarrow 1_{R[U^{-1}]\text{-mod}}$

Lemma. The unit and counit are natural

Here  $F: \mathcal{C} \rightarrow \mathcal{D}$   $G: \mathcal{D} \rightarrow \mathcal{C}$   
and there is a bijection

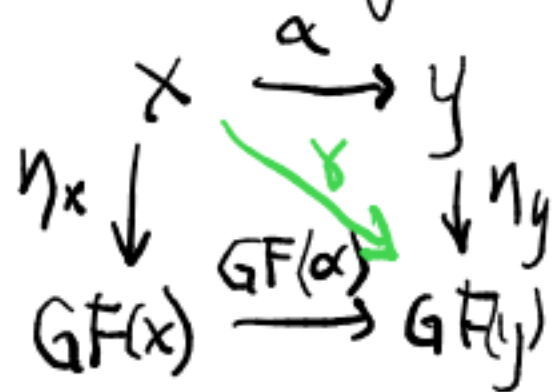
$f: \text{Hom}_{\mathcal{D}}(F(x), y) \rightarrow \text{Hom}_{\mathcal{C}}(x, G(y))$   
natural in both  $x$  and  $y$ .

To get the unit take  $y = F(x)$

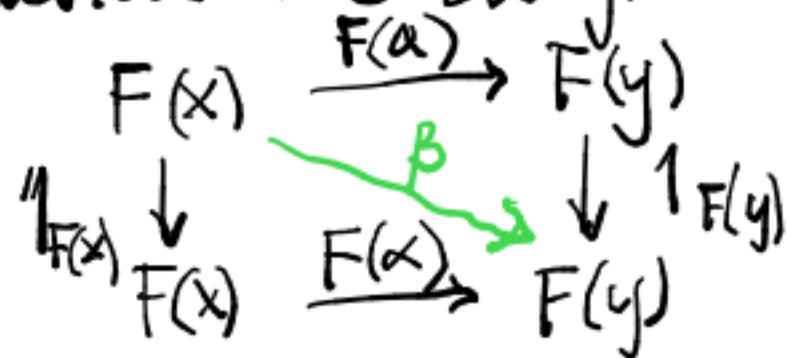
$f: \text{Hom}(F(x), F(x)) \rightarrow \text{Hom}(x, GF(x))$

$\eta_x = f(1_{F(x)}) : x \rightarrow GF(x)$

To show that  $\eta : 1_{\mathcal{C}} \rightarrow GF$   
is natural we verify  $\forall \alpha : x \rightarrow y$   
in  $\mathcal{D}$  the diagram commutes:



Consider the diagram



The vertical arrows correspond to  $\eta_x, \eta_y$  under  $f$  or  $f^{-1}$

The diagram commutes!

We use naturality of  $f$  in each variable to deduce: first diagram commutes.

We get:  $\eta_y \alpha = f(\beta)$  from the top  $\Delta$  in the second diagram.

b/c  $f(1_{F(y)}) = \eta_y$

$f(1_{F(y)} F(\alpha)) = \alpha^*(\eta_y) = \eta_y \circ \alpha$

Similarly:  $f(F(\alpha) 1_{F(x)}) = GF(\alpha) \circ \eta_x$

B/c  $1_{F(y)} F(\alpha) = F(\alpha) 1_{F(x)}$ , these are equal  $\square$



The unit and counit determine the adjunction.

Proposition. Let  $F : C \rightarrow D$  be left adjoint to  $G : D \rightarrow C$ , so that we have a bijection

$$f : \text{Hom}_D(Fx, y) \rightarrow \text{Hom}_C(x, Gy), \quad \forall x, y$$

Let  $\eta$  and  $\epsilon$  be the unit and counit of the adjunction.

If  $u : F(x) \rightarrow y$  in  $C$  then  $f(u)$  is the composite

$$x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(u)} G(y)$$

If  $v : x \rightarrow G(y)$  in  $X$  then  $f^{-1}(v)$  is the composite

$$F(x) \xrightarrow{F(v)} FG(y) \xrightarrow{\epsilon_y} y$$

Question: Do you think you could prove this? (in the next 5 mins?)

Yes

No ✓

Proof

Write  $u = u \circ 1_{F(x)}$

$$\text{Now } f(u) = G(u) \circ f(1_{F(x)})$$

by Monday's pre-class warm up.  
(naturality of  $f$  in 2nd variable).  
This proves

Write  $v = 1_{G(y)} \circ v$

to get  $f^{-1}(v) = F^{-1}(1_{G(y)})$ .

# Pre-class Warm-up!!

Let  $F : C \rightarrow D$  be left adjoint to  $G : D \rightarrow C$ .  
What does the unit of the adjunction look like? Is it

A  $\eta : 1 \rightarrow GF$

B  $\eta : 1 \rightarrow FG$

C  $\eta : GF \rightarrow 1$

D  $\eta : FG \rightarrow 1$

E  $\eta : CD \rightarrow \mathbb{1}$

F  $\eta : \mathbb{1} \rightarrow DC$

## The triangular identities

Proposition. Let  $F : C \rightarrow D$  be left adjoint to  $G : D \rightarrow C$ .

Let  $\eta$  and  $\epsilon$  be the unit and counit of the adjunction.

Then the following two triangles commute.

$$\begin{array}{ccc}
 F & \xrightarrow{1_F} & F \\
 F\eta \searrow & & \nearrow \epsilon_F \\
 & FG F & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xrightarrow{1_G} & G \\
 \eta_G \searrow & & \nearrow \epsilon_G \\
 & GFG & 
 \end{array}$$

Interpretation: The morphisms are natural transformations. Given  $x \in C$  we have

$\eta_x : x \rightarrow GF(x)$  so  $F\eta_x : F(x) \rightarrow FG F(x)$  is part of a nat. transf.  $F\eta$ .

Also  $\epsilon_y : FG(y) \rightarrow y$ . Take  $F(x) = y$  to get  $\epsilon_{F(x)} : FG F(x) \rightarrow F(x)$ .  $\epsilon_F$  is this nat. transf.

$$\forall x, \quad F(x) \xrightarrow{F\eta_x} FG F(x) \xrightarrow{\epsilon_{F(x)}} F(x)$$

is  $1_{F(x)} : F(x) \rightarrow F(x)$  in  $\mathcal{C}$ .

The adjunction is  $f : \text{Hom}(F(x), y) \rightarrow \text{Hom}(x, G(y))$

$$f(x) = x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y)$$

$$FG(1_{F(x)}) = 1_{FG F(x)} \text{ b/c } H(1_z) = 1_{H(z)}$$

Proof Given a morphism  $\alpha : F(x) \rightarrow y$  in  $\mathcal{C}$  the corresponding morphism in  $\mathcal{D}$  is the composite  $x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y)$  (\*)

Given a morphism  $\beta : x \rightarrow G(y)$  the corresponding morphism in  $\mathcal{C}$  is  $F(\beta) : F(x) \rightarrow FG(y) \xrightarrow{\epsilon_y} y$

$$\text{Take } \beta \text{ to be } (*) : F(x) \xrightarrow{F(\eta_x)} FG F(x) \xrightarrow{FG(\alpha)} FG(y) \xrightarrow{\epsilon_y} y$$

Now take  $y = F(x)$  and  $\alpha = 1_{F(x)}$ . Then  $FG F(x) \xrightarrow{FG(\alpha)} FG(y)$  is  $FG F(x) \xrightarrow{FG(1_{F(x)})} FG F(x)$

$$\text{We get } 1_{F(x)} = F(x) \xrightarrow{F(\eta_x)} FG F(x) \xrightarrow{\epsilon_{F(x)}} F(x) = f^{-1} f(1_{F(x)})$$

Theorem. Let  $F: \mathcal{C} \rightarrow \mathcal{D}$ ,  $G: \mathcal{D} \rightarrow \mathcal{C}$   
 be functors, let  
 $\eta: 1_{\mathcal{C}} \rightarrow GF$ ,  $\epsilon: FG \rightarrow 1_{\mathcal{D}}$   
 be natural transformations so  
 the triangle identities are satisfied.

Then the mapping

$$f: \text{Hom}_{\mathcal{D}}(F(x), y) \rightarrow \text{Hom}_{\mathcal{C}}(x, G(y))$$

given by

$$(\alpha: F(x) \rightarrow y) \mapsto \left( x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y) \right)$$

is a bijection natural in both variables  
 i.e.  $F$  is left adjoint to  $G$  with  
 unit  $\eta$ , counit  $\epsilon$ .

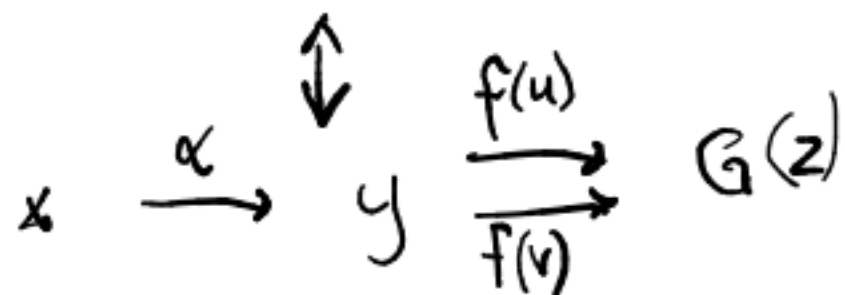


Proposition. A left adjoint preserves epimorphisms. A right adjoint preserves monomorphisms.

$x \xrightarrow{\alpha} y$  is epi  $\Leftrightarrow$  whenever  $x \xrightarrow{\alpha} y \xrightarrow[u]{v} z$  has  $u\alpha = v\alpha$  then  $u=v$ .

Proof. Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be left adjoint to  $G: \mathcal{D} \rightarrow \mathcal{C}$ , suppose  $x \xrightarrow{\alpha} y$  is epi in  $\mathcal{C}$ . To check  $F(x) \xrightarrow{F(\alpha)} F(y)$  is epi in  $\mathcal{D}$  consider  $u, v: F(y) \rightarrow z$

so that  $u F(\alpha) = v F(\alpha)$ .

$$F(x) \xrightarrow{F(\alpha)} F(y) \xrightarrow[u]{v} z$$


so that  $\left. \begin{array}{l} u \leftrightarrow f(u) \\ v \leftrightarrow f(v) \\ u F(\alpha) \leftrightarrow f(u)\alpha \\ v F(\alpha) \leftrightarrow f(v)\alpha \end{array} \right\} \text{by naturality of } f$

$$\begin{aligned}
 u F(\alpha) = v F(\alpha) &\Rightarrow F(u)\alpha = f(v)\alpha \\
 \Rightarrow f(u) = f(v) &\quad (\alpha \text{ epi}) \\
 \Rightarrow u = v &\quad (f \text{ is a bijection}) \\
 \Rightarrow F(\alpha) \text{ is epi.} &
 \end{aligned}$$

□

Application:  $M \otimes -$  is left adjoint to  $\text{Hom}(M, -)$ .  
 so  $M \otimes -$  sends epis to epis  
 i.e. is right exact.

Proposition. A left adjoint of a functor that preserves epimorphisms (and monos) sends projectives to projectives. Similarly, a right adjoint ... injectives.

A left adjoint of an exact functor between module categories

sends projectives to projectives.

Example. If  $H$  is a subgroup of  $G$ ,

$$\text{Res} : \mathbb{R}G\text{-mod} \rightarrow \mathbb{R}H\text{-mod}$$

has left adjoint  $\text{Ind}(V) = \mathbb{R}G \otimes_{\mathbb{R}H} (V)$

$\mathbb{R}$  is ~~is~~ exact so  $\text{Ind}(\text{proj}) = \text{proj}$ .

$\text{Ind}$  is also the right adjoint here

so  $\text{Ind}(\text{inj}) = \text{injective}$

## Representable functors