Category Theory
Eisenbud's Appendix 5 has the right topics
but is brief with a shortage of examples.

Definition. A category C is the
specification of

1. A class of things called 'objects'
$x \in Ob(\mathcal{C})$ means x is an object of $\mathcal{C}$.
2. For each pair $x, y \in Ob(\mathcal{C})$ we have
a set $Hom_{\mathcal{C}}(x,y)$ of things called
morphisms.
3. A rule of composition
$$Hom(y,z) \times Hom(x,y) \longrightarrow Hom(x,z)$$
$$(g\ ,\ f) \longmapsto gf \ (\text{or } g \circ f)$$
so that

a. $(hg)f = h(gf)$ always, whenever it is defined.

i. For all $x \in Ob(\mathcal{C})$, there exists
a morphism $1_x : x \to x$ so that
$f 1_x = f \ \forall f : x \to y$, $1_x j = j \ \forall j : w \to x$

Morphism notation If $f \in Hom_{\mathcal{C}}(x,y)$
we write $f : x \to y$ to denote this.
x is the 'domain' of f, y is the 'codomain' or 'target' of f.

Examples
1. **Set** = category with objects = sets
morphisms = maps of sets.

Top = category of topological spaces
morphisms = continuous maps

Group: morphisms = group homoms

R-mod: Objects are R-modules
Morphisms are R-module homomorphisms.

2. A poset P may be regarded as
a category $\mathcal{P}$ with $Ob(\mathcal{P})$ = elements
of P, $\exists$ unique morphism $x \to y$
$\Leftrightarrow x \le y$ in P.

Pre-class Warm-up!!!

Suppose f : M -> N is a homomorphism of abelian groups. Which of the following conditions necessarily implies that f is one-to-one?

A. For all pairs of homomorphisms g, h : L -> M, if fg = fh then g = h.

B. For all pairs of homomorphisms g, h : N -> Q, if gf = hf then g = h.

C. Neither of the above.

B $\Leftarrow$ f is onto.

A. $L \underset{h}{\overset{g}{\rightrightarrows}} M \xrightarrow{f} N$ $fg = fh \Rightarrow g = h$

Proposition A $\Leftrightarrow$ f is 1-1.

Proof "A $\Rightarrow$ 1-1" If f is not 1-1 then $\ker f \neq 0$. Take $L = \ker f$, $g : L \to M$ is inclusion, $h : L \to M$ is zero. Then $fg = fh = 0$ but $g \neq h$ so A fails.

"1-1 $\Rightarrow$ A" 1-1 $\Leftrightarrow$ $\ker f = 0$

If $fg = fh$ then $\forall x \in L$, $fg(x) = fh(x)$. Thus $g(x) = h(x)$ b/c f is 1-1, so $g = h$.

Definition.

$\Leftrightarrow \exists$ a morphism $g : Y \to X$ so that $gf = 1_X$ and $fg = 1_Y$.

The following is not equivalent to f being an isomorphism.

or $\Leftrightarrow$ f is 1-1 and onto ?

$\Updownarrow$

A, say 'f is a monomorphism'

I just suggested it for the purposes of discussion.

B say 'f is an epimorphism'

Question. Why do we take this definition of

More examples: a group, a monoid

Given a group G we may construct a category $\mathcal{G}$ with only one object $*$ and where $\mathrm{Hom}(*,*) = G$

composition := multiplication in G!

If M is a monoid we construct a category $\mathcal{M}$ with one object $*$ $\mathrm{Hom}(*,*) = M$.

More examples: weird categories.

Free categories

Let Q be a directed graph (or quiver)

$$\begin{tikzcd} x \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & y \arrow[r, "\gamma"] & z \end{tikzcd}$$

Construct a category $F(Q)$ where

$Ob(F(Q))$ = vertices of Q

Morphisms = all possible words in the edges of Q where the end of a symbol = start of next.

Example $Ob = \{x, y, z\}$

Morphisms $= \{1_x, 1_y, 1_z, \alpha, \beta, \gamma, \alpha^2, \beta\alpha, \gamma\beta, \alpha^3, \beta\alpha^2, \gamma\beta\alpha, \dots\}$

FI = the category with Objects = Finite sets,

Schematic description of a category with $Ob(\mathcal{C}) = \{x, y\}$

$$\begin{tikzcd} x \arrow[loop left, "C_4"] \arrow[r, "C_4"] & y \arrow[loop right, "C_4"] \end{tikzcd}$$

4 morphisms $x \to y$, $\underset{\underset{Hom_{\mathcal{C}}(x,x)}{\shortparallel}}{End(x)} = C_4$

$End(y) = C_4$

Question:

How many morphisms does $F(Q)$ have when

$Q =$

$$\begin{tikzcd} x \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & z \\ & y \arrow[ur, "\gamma"'] & \end{tikzcd}$$

A 3

B 4

C 5

D 6

E 7

F 8

G Infinitely many.

$$\begin{tikzcd} \bullet \arrow[loop left, "C_4"] \arrow[r, "C_4"] & \bullet \arrow[loop right, "C_4"] \end{tikzcd}$$

$\{1_y, a_y, \dots\}$

$C_4 = \{1_x, a_x, a_x^2, a_x^3\}$

$\{1, a, a^2, a^3\}$

Composition

$$a^2 \circ a_x = a^3$$

$$a_x^2 \circ a_x = a_x^3$$

Pre-class Warm-up!

How many morphisms are there in the free category generated by the quiver

$$
\begin{tikzcd}
w \arrow[r, "\alpha"] & x \arrow[r, "\beta"] & y & z \arrow[l, "\gamma"']
\end{tikzcd}
$$

A 3

B 4

C 7

D 8

Constructions.

- The product of two categories $\mathcal{C}$, $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with objects (c,d), $c \in Ob(\mathcal{C})$, $d \in Ob(\mathcal{D})$, morphisms $(c,d) \xrightarrow{(f,g)} (c',d')$ where $f : c \to c'$ in $\mathcal{C}$, $g : d \to d'$ in $\mathcal{D}$.

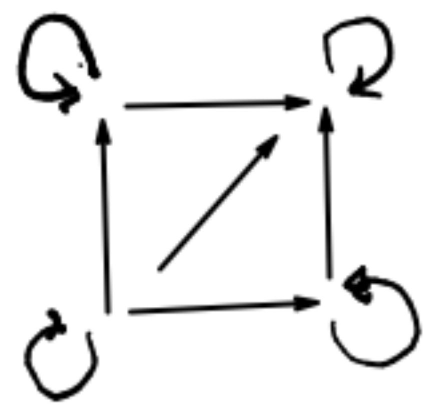
- If $\mathcal{C}$ is a category, the opposite category is $\mathcal{C}^{op}$ with $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and morphisms $\bar{\alpha}$ where α is a morphism in $\mathcal{C}$. If $\alpha : x \to y$ then $\bar{\alpha} : y \to x$. $\bar{\beta} : z \to y$, $\beta : y \to z$

$$\bar{\alpha}\bar{\beta} := \overline{\beta\alpha}$$

Question: Let I be the poset $0 \xrightarrow{\alpha} 1$

How many morphisms does I x I have?

A 1
B 2
C 4
D 6
E 8
F 9 ✓



Functors

Definition. $\mathcal{C}, \mathcal{D}$ are categories. the specification of

- $\forall\ x \in Ob(\mathcal{C})$, an object $T(x) \in Ob(\mathcal{D})$
- $\forall$ morphisms $f : x \to y$ in $\mathcal{C}$, a morphism $T(f) : T(x) \to T(y)$ in $\mathcal{D}$

so that

1. $T(fg) = T(f)\,T(g)$
2. $T(1_x) = 1_{T(x)}$ $\quad \forall x \in Ob(\mathcal{C})$

Question: did we need to put in $T(1) = 1$ always, or did it follow from the other axioms

Examples $F : AbGroups \to Groups$

Inclusion $F(A) = A$, $F(f) = f$

Forgetful functor like $F : Groups \to Set$

$F(G) = G$ regarded as a set

or $k[x]$-mod $\to$ k-mod = vector spaces over k.

If G and H are groups we get categories $\mathcal{G}, \mathcal{H}$. A functor $F : \mathcal{G} \to \mathcal{H}$ is 'the same thing as' a group homomorphism $G \to H$.

If P and Q are posets a functor $F : P \to Q$ is 'the same thing as' an order preserving map $P \to Q$.

If X is a set let $R(X)$ = free R-module with X as a basis.

R is a commutative ring

$R(-)$ is a functor $Set \to R$-mod

$F(-) : Set \to Group$ is a functor. free group generated by X

If M is a right R-module, L is a left R-module, we have functors

$M \otimes - : R\text{-mod} \to \text{Ab Group}$

$\mathrm{Hom}_R(L, -) : R\text{-mod} \to \text{Ab Group}$

R-mod = category of left R-modules
mod-R = right

These are covariant functors. The functor

Hom_R(- , L) : R-mod -> Abelian groups

F is covariant means $F(\alpha\beta) = F(\alpha)F(\beta)$
$\mathcal{C} \to \mathcal{D}$

A contravariant functor has the same definition except $F(\alpha\beta) = F(\beta)F(\alpha)$

It is the same thing as a (covariant) functor $\mathcal{C}^{op} \to \mathcal{D}$.

If G is a group (or a monoid) a functor F : G -> R-mod

A repn of G over R is a homomorphism $G \to GL(V)$ for some R-module V.

$F : G \to R\text{-mod}$ is

$F(*) = V$

$\forall\ g : * \to *$, $F(g) : V \to V$ is an R-module homomorphism.

$F(g g^{-1}) = F(1_*) = 1_V : V \to V$

$= F(g) F(g^{-1})$

$F(g^{-1}) = F(g)^{-1}$

Definition. A category C is small if Ob(C) is a set.

Example:
SCat is the category of small categories, whose objects are small categories, and whose morphisms are functors.

$\rho : G \to GL(V)$ $\quad \sigma : G \to GL(W)$

$\rho \to \sigma$

Morphisms in the category Reps of G are R-linear maps $\theta : V \to W$ so that $\theta(\rho(g)(v)) = \sigma(g)(\theta(v))$.

i.e the square

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\theta"'] & V \arrow[d, "\theta"] \\ W \arrow[r, "\sigma(g)"] & W \end{tikzcd}$$

commutes $\forall g \in G$.

This defines a homomorphism of rep'ns $\rho \to \sigma$

The functors between the category of representations of a group G and the category of RG-modules.

Reps of G are the same thing as RG-modules

RG = group ring of G
= free R-module with elts of G as basis

Multn in RG is determined by group multiplication.

We get functors

Reps of G $\rightleftarrows$ RG-mod

Given $\rho : G \to GL(V)$ get an RG-module V

$$\Big(\sum_{g\in G} a_g g\Big)\cdot v := \sum_{g\in G} a_g\, \rho(g)(v)$$

Pre-class Warm-up!

Let F : C -> D be a functor. Which of the following do you think means that F is an isomorphism of categories?

A For all objects x of C , x is isomorphic to F(x).

B For all pairs of objects x, y of C, F induces an isomorphism Hom_C(x,y) ≈ Hom_D(F(x),F(y))

C There is a functor G : D -> C so that FG = 1_D and GF = 1_C. ✓

D None of the above.

Example. The categorys

$$\text{Reps of } G \text{ over } R \underset{G}{\overset{F}{\rightleftarrows}} RG\text{-mod}$$

are isomorphic.

Given $\rho: G \to GL(V)$ we get an RG-module $F(\rho) = V$, $\sum a_g g \cdot v := \sum a_g \rho(g)(v)$

Given an RG-module W we get a homom. $G(W): G \to GL(W)$

$g \longmapsto (w \mapsto g \cdot v)$.

$GF(\rho) = \rho$

$FG(V) = F(G \to GL(V))$

$= V$ with original module action of RG.

If Q is a directed graph (= a quiver), a representation of Q over R is the specification of

- for each vertex x of Q, an R-module M_x
- for each arrow $x \xrightarrow{\alpha} y$ in Q an R-module homomorphism

$$M_x \xrightarrow{M(\alpha)} M_y$$

Example: $Q = \begin{tikzcd} \bullet x \arrow[loop left, "\alpha"] \end{tikzcd}$ a repn M is an R-module M_x with an R-linear map $M(\alpha): M_x \to M_x$.

This is the same thing as a $R[t]$-module where t acts via α.

It is the same thing as a functor $F(Q)$ -> R-mod

A homomorphism of quiver representations the specification of, for each vertex x of Q $= x \in \mathrm{Ob}\, F(Q)$ an R-module homomorphism

$$\begin{tikzcd} F_1(x) \arrow[r, "\mu_x"] \arrow[d, equal] & F_2(x) \arrow[d, equal] \\ M_{1x} & M_{2x} \end{tikzcd}$$

so that every diagram commutes

$$\begin{tikzcd} F_1(x) \arrow[r, "\mu_x"] \arrow[d, "F_1(\alpha)"'] & F_2(x) \arrow[d, "F_2(\alpha)"] \\ F_1(y) \arrow[r, "\mu_y"] & F_2(y) \end{tikzcd}$$

($\forall$ morphisms α in $F(Q)$)

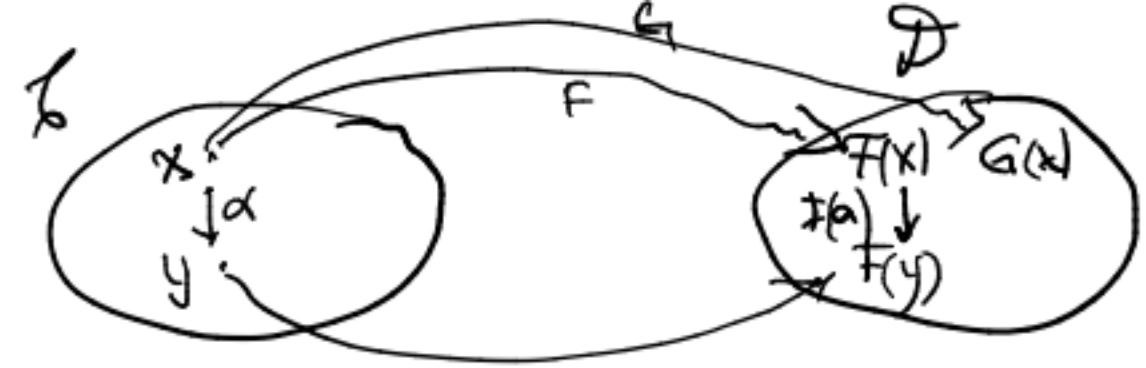
Natural transformations

These are morphisms between functors, comparable to the notion of homotopy between maps of topological spaces.

Definition. Let F , G : C -> D by functors.
A natural transformation
$\mu : F \to G$ is the specification of, for each object $x \in Ob(\mathcal{C})$ a morphism $\mu_x : F(x) \to G(x)$ in $\mathcal{D}$ so that, $\forall$ morphisms $\alpha : x \to y$ in $\mathcal{C}$ the square

$$\begin{array}{ccc} F(x) & \xrightarrow{\mu_x} & G(x) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(y) & \xrightarrow{\mu_y} & G(y) \end{array}$$

commutes.



Examples: Homomorphism of group representations and of quiver
Regarding repns of G as functors $\rho: G \to R\text{-mod}$ a homomorphism $\rho \to \sigma$ is a natural transformation. $\rho \to \sigma$

Regarding repns of a quiver Q as functors F(Q), a homomorphism of repns $F_1 \to F_2$ is a natural transformation.

Examples: Homomorphism of group representations and of quiver

Done.

SCat has
- Objects : categories
- Morphisms : functors

Fun
- Objects : functors
- Morphisms : nat transfs

In fact SCat is a 2-category.

Example: Hom(C,D) where C and D are categories.

Let $\mathcal{C}$, $\mathcal{D}$ be categories. We define a category $\mathrm{Hom}_{\mathrm{SCat}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$

with objects := functors $\mathcal{C} \to \mathcal{D}$

morphisms $F \to G$:= natural transformations $F \to G$.

Implicit: we can compose natural transfs $F \xrightarrow{\mu} G \xrightarrow{\lambda} H$

Then $(\lambda\mu)_x := \lambda_x \mu_x : F(x) \to H(x)$ in $\mathcal{D}$

There is a $1_F : F \to F$ for each functor F.

Pre-class Warm-up!!

Let M be an R-module. Is the operation that sends

$\mathrm{Hom}_R(R, M)$ to M

by sending

$\phi \longmapsto \phi(1)$

A a functor

B a natural transformation

C a category

D none of the above.

We have functors

$\mathrm{Hom}_R(R,-) : R\text{-mod} \longrightarrow R\text{-mod}$

$1 : R\text{-mod} \longrightarrow R\text{-mod}.$

$\eta : \mathrm{Hom}(R,-) \longrightarrow 1$

specified by

$\eta_M : \mathrm{Hom}(R,M) \longrightarrow 1(M) = M$

$\eta_M(\phi) = \phi(1).$

is a natural transformation

If $\alpha : M \to N$ then

$$\begin{array}{ccc} \mathrm{Hom}(R,M) & \xrightarrow{\eta_M} & M \\ {\scriptstyle \mathrm{Hom}_R(R,\alpha)} \downarrow & & \downarrow {\scriptstyle \alpha} \\ \mathrm{Hom}(R,N) & \xrightarrow{\eta_N} & N \end{array}$$

commutes.

The double dual.

Let V be a finite dimensional vector space over a field k. Let V^* $=_k$ Hom (V,k) be the vector space dual.

Question: Is the operation that sends V to V^*

A a functor?

B a natural transformation?

$$\begin{array}{ccc} k\text{-mod} & \longrightarrow & k\text{-mod} \\ V & \longmapsto & V^* \end{array}$$

is a contravariant functor i.e. a functor $k\text{-mod}^{op} \longrightarrow k\text{-mod}$

$F(V) = V^{**}$

$F : k\text{-mod} \to k\text{-mod}.$

$V \not\cong V^{**}$

We have natural transformation

$\eta : 1 \longrightarrow (-)^{**}$

$\eta_V : V \longrightarrow V^{**}$

$\eta_V(w) = (f \longmapsto f(w))$ for $w \in V$.

Also there is a nat. transfn $\Theta : (-)^{**} \longrightarrow V$ if V is finite dimensional. On the category where $\dim V < \infty \Rightarrow 1$ and $(-)^{**}$ are naturally isomorphic functors.

Question:
Is the functor V -> V^* naturally isomorphic to the identity functor on finite-dimensional vector spaces?

A Yes

B No ✓

1 is covariant : $k\text{-mod} \to k\text{-mod}$

$(-)^*$ is contravariant.

Example.
Given a finite set X and a ring R we may construct the free R-module R^X on X. Let N be an R-module.
Consider the two functors Set -> Set specified by
if Y is a set then

F(Y) = $\mathrm{Hom}_{Set}(Y, N)$

G(Y) = $\mathrm{Hom}_R(R^Y, N)$.

Any map of sets $\phi : Y \to N$ extends uniquely to an R-module homomorphism $\eta_Y(\phi) : R^Y \to N$. Here η_Y is a bijection of sets

$\eta_Y : F(Y) \to G(Y)$.

η is a natural isomorphism $F \cong G$

Pre-class Warm-up

Given a finite set X and a ring R we may construct the free R-module R^X on X.
Let f : M -> N be a homomorphism of R-modules.
Fixing X, consider the two functors R-mod -> Set specified by

$F(M) = \mathrm{Hom}_{Set}(X, M)$

$G(M) = \mathrm{Hom}_{R\text{-mod}}(R^X, M)$

On Wednesday we considered a bijection of sets $\theta_M : F(M) \to G(M)$.

Does the square commute?

$$\begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, leftrightarrow, "\theta_M"'] & F(N) \arrow[d, leftrightarrow, "\theta_N"] \\ G(M) \arrow[r, "G(f)"] & G(N) \end{tikzcd}$$

A Yes

If $\alpha : X \to M$ is a map of sets then
$F(f)(\alpha) := f\alpha$.
$G(f)$ is also postcomposition with f

Natural isomorphism, equivalence of categories.

If $F, G : \mathcal{C} \to \mathcal{D}$ are two functors a natural isomorphism is a natural transformation $\eta : F \to G$ for which $\exists$ a nat. transfn. $\theta : G \to F$ so that $\theta\eta = 1_F$, $\eta\theta = 1_G$.

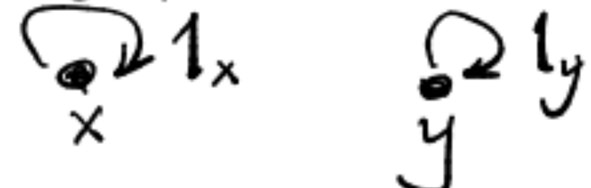
(Exercise: $\Leftrightarrow$ $\eta_x : F(x) \to G(x)$ is an isomorphism in $\mathcal{D}$, $\forall\, x \in Ob(\mathcal{C})$.)

Definition: Categories $\mathcal{C}, \mathcal{D}$ are equivalent $\Leftrightarrow$ $\exists$ functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ plus natural isomorphisms $\theta : GF \xrightarrow{\simeq} 1_{\mathcal{C}}$ and $\psi : FG \xrightarrow{\simeq} 1_{\mathcal{D}}$.

If $\mathcal{C}, \mathcal{D}$ are isomorphic as categories they are equivalent

Example.
Let C be the category with two objects x and y, and with only the identity morphisms. Let k be a field.
We compare
the category of functors C -> k-mod
to the category of k x k -modules.

$\mathcal{C}$ is



A functor $\mathcal{C} \to k\text{-mod}$ is the specification of two k-vector spaces

Get a functor $F : Fun(\mathcal{C}, k\text{-mod}) \to k\times k\text{-mod}$

$k \times k = \{(u,v) \mid u, v \in k\}$

$F(A) =$ the $k\times k$-module $A(x) \oplus A(y)$

where $(u,v)\cdot(a,b) := (ua, vb)$

We define $G : k\times k\text{-mod} \to Fun(\mathcal{C}, k\text{-mod})$

$$G(B) = \begin{cases} x \longmapsto (1,0)\cdot B \\ y \longmapsto (0,1)\cdot B \end{cases}$$

$F(A) = A(x) \oplus A(y) \in k \times k\text{-mod}$

$G(B) = \begin{cases} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{cases}$

$\in \mathrm{Fun}(\mathcal{C}, k\text{-mod})$.

$F : \mathrm{Fun}(\mathcal{C}, k) \rightleftarrows k \times k\text{-mod} : G$

$GF(A) = G(A(x) \oplus A(y))$

$= \begin{cases} x \longmapsto (1,0)(A(x) \oplus A(y)) \\ \qquad\qquad \overset{\shortparallel}{(A(x) \oplus 0)} \\ y \longmapsto 0 \oplus A(y). \end{cases}$

looks like A but is different.

$FG(B) = F\begin{pmatrix} x \longmapsto (1,0)B \\ y \longmapsto (0,1)B \end{pmatrix}$

$= (1,0)B \oplus (0,1)B$.

$=$ set of pairs $((1,0)a, (0,1)b)$
$a, b \in B$.

looks like B but is different

F, G are not inverse isomorphisms

However $GF \cong 1_{\mathrm{Fun}(\mathcal{C}, k\text{-mod})}$

$FG \cong 1_{k \times k\text{-mod}}$

Define $\theta : 1_{\mathrm{Fun}(\ )} \longrightarrow GF$

$\theta_A : A \to GF(A)$ is morphism of functors $=$ a nat'l transfn.

$\theta_{A,x} : A(x) \longrightarrow GF(A)(x) = (1,0)(A(x) \oplus A(y))$

$a \longmapsto (a, 0) \in A(x) \oplus A(y)$

Similarly we define $\psi : 1_{k \times k\text{-mod}} \to FG$.

$\mathrm{Fun}(\mathcal{G}, k\text{-mod})$ and

$k \times k$-mod

might be isomorphic categories.

It's a job to show this (if true)

However the categories are equivalent. This is easier to show.

Speak of two categories being equivalent rather than isomorphic.

Reprs of $\bullet \quad \bullet = Q$

'are the same as'

$k \times k$-modules.

$$F(Q) = 1_x \circlearrowright \bullet_x \qquad \bullet_y \circlearrowleft 1_y$$

Example.

Let $C = $
$$\begin{tikzcd} \underset{x}{\bullet} \arrow[loop right, "1_x"] \end{tikzcd}$$

Let $D = $
$$\begin{tikzcd} \underset{x}{\bullet} \arrow[loop left, "1_x"] \arrow[r, shift left, "\alpha"] & \underset{y}{\bullet} \arrow[l, shift left, "\beta"] \arrow[loop right, "1_y"] \end{tikzcd}$$

where $\beta\alpha = 1_x$ $\quad \alpha\beta = 1_y$

We show $\mathcal{C} \simeq \mathcal{D}$ are equivalent but $\mathcal{C} \not\cong \mathcal{D}$ (not isomorphic).

We have

$$\begin{aligned} inc : \mathcal{C} &\longrightarrow \mathcal{D} \\ x &\longmapsto x \\ 1_x &\longmapsto 1_x \end{aligned}$$

and

$$\begin{aligned} F : \mathcal{D} &\longrightarrow \mathcal{C} \\ x &\longmapsto x \\ y &\longmapsto x \\ 1_x, 1_y, \alpha, \beta &\longmapsto 1_x . \end{aligned}$$

$$F \circ inc = 1_{\mathcal{C}} .$$

$$inc \circ F \neq 1_{\mathcal{D}} .$$

Define $\theta : 1_{\mathcal{D}} \longrightarrow inc \circ F$

$\theta_x : 1_{\mathcal{D}}(x) = x \longmapsto inc \circ F(x) = x$

$\theta_y : \quad y \longmapsto x$

$\theta_x := 1_x \quad \theta_y := \alpha .$

Check this a nat. transformation

It is iso (1_x, α are isomorphisms)

$\mathcal{C} \simeq \mathcal{D}$.

Pre-class Warm-up!!!

Let U, V : C -> D be functors. Suppose for each object x of C we have a map of sets (where z is some object of D)

$$f_x : \mathrm{Hom}_{\mathcal{D}}(U(x), z) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(V(x), z)$$

Conditions A and B below both mean that f is natural with respect to x.
Which of A and B fits better with your understanding?

A For all morphisms g : x -> y in C the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(U(x), z) \arrow[d, "f_x"'] & \mathrm{Hom}_{\mathcal{D}}(U(y), z) \arrow[l, "U(g)^*"'] \arrow[d, "f_y"] \\ \mathrm{Hom}_{\mathcal{D}}(V(x), z) & \mathrm{Hom}_{\mathcal{D}}(V(y), z) \arrow[l, "V(g)^*"] \end{tikzcd}$$

B For all morphisms g : x -> y in C , whenever the next triangle commutes

$$\begin{tikzcd} U(x) \arrow[rr, "U(g)"] \arrow[dr, "r"'] & & U(y) \arrow[dl, "s"] \\ & z & \end{tikzcd}$$

the following triangle commutes.

$$\begin{tikzcd} V(x) \arrow[rr, "V(g)"] \arrow[dr, "f_x(r)"'] & & V(y) \arrow[dl, "f_y(s)"] \\ & z & \end{tikzcd}$$

Which do you prefer?

A Assume A. Show the second Δ in B commutes if the first does

B If the first commutes then $r = U(g)^*(s)$

so $f_x\, U(g)^*(s) = V(g)^* f_y(s)$

$f_x(r) = f_y(s) \cdot V(g)$. The second Δ commutes.

Adjoints

Definition.
Let $F : C \to D$ and $G : D \to C$ be functors. We say F is the left adjoint of G and G is the right adjoint of F if there is a bijection (called the adjunction)

$$\mathrm{Hom}_{\mathcal{D}}(F(x), y) \to \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

natural in both x and y.

Examples. 1. $G : R\text{-mod} \to \mathrm{Set}$
$M \longmapsto M$ as a set.

$F : \mathrm{Set} \to R\text{-mod}$, $X \mapsto R^X$ = free module with X as basis.

$$\mathrm{Hom}_{R\text{-mod}}(R^X, M) \longleftrightarrow \mathrm{Hom}_{\mathrm{Set}}(X, M)$$

is a bijection natural in both X and M.

$X \longmapsto R^X$ is left adjoint to the forgetful functor $R\text{-mod} \to \mathrm{Set}$.

2. Let R, S be rings and ${}_S M_R$ an (S,R)-bimodule.

Then $\mathrm{Hom}_S(M \otimes_R N, L)$

$$\cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, L))$$

where N is a left R-module, L is a left S-module, naturally in both N and L.

Here $F = M \otimes_R - : R\text{-mod} \to S\text{-mod}$
$G = \mathrm{Hom}_S(M, -) : S\text{-mod} \to R\text{-mod}$
and F is left adjoint to G.

Special case: R is a subring of S

$M = {}_S S_R$

Then $S \otimes_R N : R\text{-mod} \to S\text{-mod}$

is left adjoint to

$\operatorname{Hom}_S(S_R, L) : S\text{-mod} \to R\text{-mod}$

$\cong$

L as an R-mod.

$S \otimes_R -$ is left adjoint to

restriction: $S\text{-mod} \to R\text{-mod}$.

The unit and counit of an adjunct ion.

Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then for $x \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{D}}(F(x), F(x)) \longleftrightarrow \mathrm{Hom}_{\mathcal{C}}(x, GF(x))$$

so

$$1_{F(x)} \longleftrightarrow \eta_x : x \to GF(x).$$

Write $\boxed{\eta : 1_{\mathcal{C}} \to GF}$

Also for $y \in \mathcal{D}$

$$\mathrm{Hom}_{\mathcal{C}}(G(y), G(y)) \longleftrightarrow \mathrm{Hom}_{\mathcal{D}}(FG(y), y)$$

$$1_{G(y)} \longleftrightarrow \epsilon_y : FG(y) \to y$$

Write $\boxed{\epsilon : FG \to 1_{\mathcal{D}}}$

η is the unit of the adjunction

ϵ is the counit of the adjunction.

Examples.

1. $F(X) = R^X \qquad G(M) = M$ as a set.

$GF(X) = R^X$ as a set.

$$\eta : 1_{\mathrm{set}} \longrightarrow GF(X)$$

$\eta_X : X \longrightarrow R^X$ embeds X as the basis of R^X

$$\epsilon : FG(M) = R^M \to M$$

is the homomorphism determined by sending basis element $m \in R^M$ to $m \in M$.

Pre-class Warm-up!!

Let U be a multiplicatively closed subset of a commutative ring R and let M be an R-module. Given that the functor
F(M) = M[U^{-1}]
is left adjoint to the inclusion functor
R[U^{-1}]-mod -> R-mod, do you think the map

$$\theta_M : M \longrightarrow M[U^{-1}]$$
$$m \longmapsto \frac{m}{1}$$

is

A the unit of the adjunction ✓

B the counit of the adjunction?

$$F : M \longmapsto M[U^{-1}]$$

$$\text{R-mod} \underset{\text{Forget} = G}{\overset{F}{\rightleftarrows}} R[U^{-1}]\text{-mod}$$

F is left adjoint to G.

Bijection.

$$\mathrm{Hom}_{R[U^{-1}]}(M[U^{-1}], N) \longleftrightarrow \mathrm{Hom}_R(M, N).$$

Unit. $\eta : 1_{\text{R-mod}} \longrightarrow GF$

Counit $\varepsilon : FG \longrightarrow 1_{R[U^{-1}]\text{-mod}}$.

Lemma. The unit and counit are natural

Here $F: \mathcal{C} \to \mathcal{D}$ $\quad G: \mathcal{D} \to \mathcal{C}$
and there is a bijection
$f: \mathrm{Hom}_{\mathcal{D}}(F(x), y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, G(y))$
natural in both x and y.

To get the unit take $y = F(x)$
$f: \mathrm{Hom}(F(x), F(x)) \longrightarrow \mathrm{Hom}(x, GF(x))$
$\eta_x = f(1_{F(x)}) : x \longrightarrow GF(x)$.

To show that $\eta: 1_{\mathcal{C}} \to GF$ is natural we verify $\forall\, \alpha: x \to y$ in $\mathcal{D}$ the diagram commutes:

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, "\eta_x"'] \arrow[dr, "\gamma"] & y \arrow[d, "\eta_y"] \\ GF(x) \arrow[r, "GF(\alpha)"'] & GF(y) \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} F(x) \arrow[r, "F(\alpha)"] \arrow[d, "1_{F(x)}"'] \arrow[dr, "\beta"] & F(y) \arrow[d, "1_{F(y)}"] \\ F(x) \arrow[r, "F(\alpha)"'] & F(y) \end{tikzcd}$$

The vertical arrows correspond to η_x, η_y under f or f^{-1}

The diagram commutes!

We use naturality of f in each variable to deduce: first diagram commutes.

We get: $\eta_y \alpha = f(\beta)$ from the top Δ in the second diagram.

b/c $f(1_{F(y)}) = \eta_y$

$f(1_{F(y)} F(\alpha)) = \alpha^*(\eta_y) = \eta_y \circ \alpha$.

Similarly: $f(F(\alpha)\, 1_{F(x)}) = GF(\alpha) \circ \eta_x$

B/c $1_{F(y)} F(\alpha) = F(\alpha)\, 1_{F(x)}$, these are equal

$\square$

The unit and counit determine the adjunction.

Proposition. Let F : C -> D be left adjoint to G : D -> C, so that we have a bijection

$$f : \mathrm{Hom}_{\mathcal{C}}(Fx, y) \to \mathrm{Hom}_{\mathcal{D}}(x, Gy), \ \forall x, y$$

Let η and ϵ be the unit and counit of the adjunction.
If u : F(x) -> y in C then f(u) is the composite

$$x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(u)} G(y)$$

If v : x -> G(y) in X then f^{-1}(v) is the composite

$$F(x) \xrightarrow{F(v)} FG(y) \xrightarrow{\epsilon_y} y$$

Question: Do you think you could prove this? (in the next 5 mins?)

Yes No ✓

Proof

Write $u = u \circ 1_{F(x)}$

Now $f(u) = G(u)_* f(1_{F(x)})$

by Monday's pre-class warm up. (naturality of f in 2nd variable).

This proves

Write $v = 1_{G(y)} \circ v$

to get $f^{-1}(v) = F(v)^* f^{-1}(1_{G(y)})$.

Pre-class Warm-up!!

Let F : C -> D be left adjoint to G : D -> C.
What does the unit of the adjunction look like? Is it

A $\eta : 1 \to GF$

B $\eta : 1 \to FG$

C $\eta : GF \to 1$

D $\eta : FG \to 1$

E $\eta : CD \to \mathbb{1}$

F $\eta : \mathbb{1} \to DC$

The triangular identities

Proposition. Let F : C -> D be left adjoint to G : D -> C.
Let η and ϵ be the unit and counit of the adjunction.
Then the following two triangles commute.

$$
\begin{tikzcd}
F \arrow[rr, "1_F"] \arrow[dr, "F\eta"'] & & F \\
& FGF \arrow[ur, "\epsilon_F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[rr, "1_G"] \arrow[dr, "\eta_G"'] & & G \\
& GFG \arrow[ur, "G\epsilon"'] &
\end{tikzcd}
$$

Interpretation: The morphisms are natural transformations. Given $x \in C$ we have $\eta_x : x \to GF(x)$ so $F\eta_x : F(x) \to FGF(x)$ is part of a nat. transf. $F\eta$. Also $\epsilon_y : FG(y) \to y$. Take $F(x) = y$ to get $\epsilon_{F(x)} : FGF(x) \to F(x)$. ϵ_F is this nat. transf.

$\forall x$, $F(x) \xrightarrow{F\eta_x} FGF(x) \xrightarrow{\epsilon_{F(x)}} F(x)$ is $1_{F(x)} : F(x) \to F(x)$ in $\mathcal{D}$.

The adjunction is $f : \mathrm{Hom}(F(x), y) \to \mathrm{Hom}(x, G(y))$

$$f(\alpha) = x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y)$$

$$FG(1_{F(x)}) = 1_{FGF(x)} \text{ b/c } H(1_z) = 1_{H(z)}$$

Proof Given a morphism $\alpha : F(x) \to y$ in $\mathcal{D}$ the corresponding morphism in $\mathcal{C}$ is the composite

$$x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y) \qquad (*)$$

Given a morphism $\beta : x \to G(y)$ the corresponding morphism in $\mathcal{D}$ is

$$f^{-1}(\beta) : F(x) \xrightarrow{F(\beta)} FG(y) \xrightarrow{\epsilon_y} y$$

Take β to be $(*)$:

$$F(x) \xrightarrow{F(\eta_x)} FGF(x) \xrightarrow{FG(\alpha)} FG(y) \xrightarrow{\epsilon_y} y$$

Now take $y = F(x)$ and $\alpha = 1_{F(x)}$. Then $FGF(x) \xrightarrow{FG(\alpha)} FG(y)$ is $FGF(x) \xrightarrow[\overset{\shortparallel}{1_{FGF(x)}}]{FG1_{F(x)}} FGF(x)$

We get $1_{F(x)} = F(x) \xrightarrow{F(\eta_x)} FGF(x) \xrightarrow{\epsilon_{F(x)}} F(X)$
$= f^{-1} f(1_{F(x)})$

Theorem. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors, let

$\eta : 1_{\mathcal{C}} \to GF$, $\epsilon : FG \to 1_{\mathcal{D}}$

be natural transformations so the triangle identities are satisfied.

Then the mapping

$$f : \mathrm{Hom}_{\mathcal{D}}(F(x), y) \to \mathrm{Hom}_{\mathcal{C}}(x, G(y))$$

given by

$$\left(\alpha : F(x) \to y\right) \longmapsto \left(x \xrightarrow{\eta_x} GF(x) \xrightarrow{G(\alpha)} G(y)\right)$$

is a bijection natural in both variables

i.e. F is left adjoint to G with unit η, counit ϵ.

Proposition. A left adjoint preserves epimorphisms. A right adjoint preserves monomorphisms.

$x \xrightarrow{\alpha} y$ is epi $\iff$ whenever $x \xrightarrow{\alpha} y \rightrightarrows^{u}_{v} z$ has $u\alpha = v\alpha$ then $u = v$.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be left adjoint to $G : \mathcal{D} \to \mathcal{C}$, suppose $x \xrightarrow{\alpha} y$ is epi in $\mathcal{C}$. To check $F(x) \xrightarrow{F(\alpha)} F(y)$ is epi in $\mathcal{D}$ consider $u, v : F(y) \to z$ so that $u F(\alpha) = v F(\alpha)$.

$$F(x) \xrightarrow{F(\alpha)} F(y) \rightrightarrows^{u}_{v} z$$

$$\Updownarrow$$

$$x \xrightarrow{\alpha} y \rightrightarrows^{f(u)}_{f(v)} G(z)$$

so that

$$\left.\begin{array}{l} u \leftrightarrow f(u) \\ v \leftrightarrow f(v) \\ uF(\alpha) \leftrightarrow f(u)\alpha \\ vF(\alpha) \leftrightarrow f(v)\alpha \end{array}\right] \text{by naturality of } f.$$

$uF(\alpha) = vF(\alpha) \Rightarrow f(u)\alpha = f(v)\alpha$

$\Rightarrow f(u) = f(v)$ (α epi)

$\Rightarrow u = v$ (f is a bijection)

$\Rightarrow F(\alpha)$ is epi.

$\square$

Application: $M \otimes -$ is left adjoint to $\mathrm{Hom}(M, -)$ so $M \otimes -$ sends epis to epis i.e. is right exact.

Proposition. A left adjoint of a functor that preserves epimorphisms (and monos) sends projectives to projectives. Similarly, a right adjoint … injectives.

A left adjoint of an exact functor between module categories sends projectives to projectives.

Example. If H is a subgroup of G, $\mathrm{Res}: RG\text{-mod} \to RH\text{-mod}$ has left adjoint $\mathrm{Ind}(V) = RG \otimes_{RH} (V)$. R is exact so $\mathrm{Ind}(\text{proj}) = \text{proj}$. Ind is also the right adjoint here so $\mathrm{Ind}(\text{inj}) = \text{injective}$.

Defn. $x \in \mathcal{C}$ is projective $\Leftrightarrow$ whenever we have morphisms

$$\begin{tikzcd} & x \arrow[d, "\beta"] \\ y \arrow[r, "\alpha"] & z \end{tikzcd}$$

with α epi, $\exists$ $\gamma : x \to y$, $\beta = \alpha\gamma$.

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Thus $\exists$ natural bijection $f : \mathrm{Hom}_{\mathcal{D}}(F(x), y) \to \mathrm{Hom}_{\mathcal{C}}(x, G(y))$. Let x be projective in $\mathcal{D}$. To test projectivity of $F(x)$ consider morphisms

$$\begin{tikzcd} & F(x) \arrow[d, "\beta"] \\ y \arrow[r, "\alpha"] & z \end{tikzcd}$$

with α epi. We construct

$$\begin{tikzcd} & x \arrow[d, "F(\beta)"] \arrow[dl, dashed, "\gamma"'] \\ G(y) \arrow[r, "G(\alpha)"] & G(z) \end{tikzcd}$$

where $G(\alpha)$ is epi. x is projective so $\exists$ $\gamma : x \to G(y)$ with $f(\beta) = G(\alpha)\gamma$. Going back the triangle

$$\begin{tikzcd} & F(x) \arrow[d, "\beta"] \arrow[dl, dashed, "f^{-1}(\gamma)"'] \\ y \arrow[r, "\alpha"] & z \end{tikzcd}$$

commutes. $F(x)$ is projective. $\square$

Question: do we understand why the last triangle commutes? It's naturality of f in 2nd variable.

A Yes B No.

Pre-class Warm-up!

Are any of the following functors R-mod -> Set naturally isomorphic?

1. The forgetful functor F(M) = M regarded as a set.

2. $\mathrm{Hom}_{R\text{-mod}}(R, -)$

$\mathrm{Hom}_{R\text{-mod}}(R, M) \cong M$

$\phi \longleftrightarrow \phi(1)$

3. $\mathrm{Hom}_{R\text{-mod}}(-, R)$

Answers:

A: 1 and 2

B: 1 and 3

C: 2 and 3

Representable functors

Definition. A functor F : C -> Set is representable there is $x \in Ob(\mathcal{C})$ so that F is naturally isomorphic to $Hom_{\mathcal{C}}(x,-)$. We say F is representable at x.

F is representable if and only if there exists x so that F is representable at x.

Examples

1. Forget : R-mod $\to$ Set is representable at R.

$Hom_{R\text{-mod}}(R,-) \cong$ Forget.

2. Take a group G, regarded as a category $\mathcal{G}$ with a single object $*$.
Functors $\mathcal{G} \to$ Set are the same thing as permutation representations of G.
The representable functor $Hom_{\mathcal{G}}(*,-)$ send $*$ to G with permutation action given by multiplication. This is the regular representation.

3. Given a monoid M we construct a category $\hat{M}$ with objects the idempotents $e=e^2$ in M

$Hom_{\hat{M}}(e,f) := fMe \subseteq M$

Composition is multiplication.

The representable functors $\hat{M} \longrightarrow \mathrm{Set}$ are the $\mathrm{Hom}_{\hat{M}}(e, -)$. At an object f we get $\mathrm{Hom}_{\hat{M}}(e, f) = fMe$.

Thus 'corresponds' to the set $\bigcup_{f \text{ idempotent}} fMe = Me$

Lemma (Yoneda's Lemma). Let x be an object of $\mathcal{C}$ and $F : \mathcal{C} \to \mathrm{Set}$ be a functor. Then

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(x,-), F)$$

bijects with $F(x)$.

Proof. Given a natural transformation

$\theta : \mathrm{Hom}(x,-) \longrightarrow F$

we get $\theta_x(1_x) \in F(x)$.

Given an element $u \in F(x)$ we construct $\psi : \mathrm{Hom}(x,-) \to F$

If $y \in \mathcal{C}$ we define $\psi_y : \mathrm{Hom}(x,y) \to F(y)$ by $\psi_y(f) := F(f)(u) \in F(y)$.

These constructions are mutually inverse.

Check this: Start with θ

Get $u = \theta_x(1_x) \in F(x)$

and $\psi_y(f) = F(f)(\theta_x(1_x))$

$= \theta_y(f)$

because naturality of θ means the square commutes

$$\begin{tikzcd} 1_x \in \mathrm{Hom}(x,x) \arrow[r, "\theta_x"] \arrow[d, "f_*"'] & F(x) \ni \theta_x(1_x) \arrow[d, "F(f)"] \\ \mathrm{Hom}(x,y) \arrow[r, "\theta_y"] & F(y) \ni F(f)(\theta_x(1_x)) \end{tikzcd}$$

$f_*(1_x) = f$ $\qquad$ $\theta_y(f) = F(f)(\theta_x(1_x))$

Thus $\theta = \psi$. The other composite is similar. $\square$

Extension of ' Yoneda's lemma to R-linear categories

Corollary. Representable functors are projective