Example: the p-adic integers (Section 7.1)

The 10-adic integers

Decimal numbers are strings

finite integers | $a_3 a_2 a_1 a_0 \bullet a_{-1} a_{-2} - - - - - -$

~~? $\bullet$ 0 0 0~~

10-adic integers are strings

$\cdots - a_3 a_2 a_1 a_0 \bullet 000$

Example

$$\begin{array}{r} \ldots 54321 \\ + \ldots 41798 \\ \hline \ldots 96119 \end{array}$$

Let $p = 2$ and work to base 2.

This was wrong

$$\begin{array}{r} 1\,?0\,?1\,?1\,?0\,?1. \\ \times \qquad 5. \\ \hline \ldots 0001. \end{array}$$

5 is a notation for 101

$5\,(\ldots 01101101) = 1$ ✗

001

$\frac{1}{5} = \ldots \overline{0110}1101.$ ✗

It should be

$\frac{1}{5} = \ldots \overline{1100}11001101.$

Pre-class Warm-up!!

In the 2-adic integers, calculate

$\ldots 1111.000 + 1.000 = 0$

A 0

B 1

C -1

D 10

E 11

$-1 = \ldots 1111.$

Fact: The 2-adic integers are a group under +

Calculate also $\ldots\overline{10}101011.000 \times 3$ (where 3 is a notation for the 2-adic integer 11.000)

$$
\begin{array}{rr}
 & \ldots 01010111 \\
\times & 11 \\
\hline
 & \ldots 10101011 \\
+ & 01010110 \\
\hline
 & \ldots 000001
\end{array}
$$

First definition of completion

Let M be an R-module and consider a filtration

$$\cdots \cdots M_i \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

filtration = chain of submodules.

We define a pseudo-distance function on M.

Let $0 < \mu < 1$. We put, for $a, b \in M$

$$d(a,b) = \begin{cases} \mu^i & \text{if } a-b \in M_i - M_{i+1} \\ 0 & \text{if } a-b \in \bigcap_{i \geq 0} M_i \end{cases}$$

Proof of ultrametic $\leq$

If $a-b \in M_i - M_{i+1}$, $b-c \in M_j - M_{j+1}$ with $\mu^i \geq \mu^j$ then $i \leq j$ so $b-c \in M_i$, $a-c = a-b+b-c \in M_i$ so $d(a,c) \leq \mu^i$. $\square$

Proposition For all a, b in M we have

$d(a,b) \geq 0$, $d(a,b) = d(b,a)$

$$d(a,c) \leq \max(d(a,b), d(b,c)) \quad \text{the ultrametric inequality}$$

$$\leq d(a,b) + d(b,c)$$

$$d(a,b) = 0 \iff a-b \in \bigcap_{i \geq 0} M_i.$$

Definition (Krull topology) We put a topology on M with basic open sets the balls $B_\epsilon(a)$, $a \in M$, $\epsilon > 0$.

Open sets = unions of these balls.

Proposition / Exercise. This topology is Hausdorff if and only if $\bigcap M_i = \{0\}$

The following is a collection of basic open sets: $a + M_i, \quad i = 0, 1, 2, \ldots \quad a \in M.$

$= B_{\mu^{i-1}}(a)$

M is a topological group under $+$

Representing elements of M as sequences

Take a set of coset representatives

$x_{i,j}$ for M_{i+1} in M_i, j varies $j = 1, 2, 3, \ldots$

Each $m \in M$ determines a list of these coset reps: Consider $i = 0$

$M_0 \supseteq M_1$. $\quad m \equiv x_{0,j_0} \pmod{M_1}$

$m - x_{0,j_0} \equiv x_{1,j_1} \pmod{M_2}$

$m - x_{0,j_0} - x_{1,j_1} \equiv x_{2,j_2} \pmod{M_3}$

We get a list $(\ldots, x_{2,j_2}, x_{1,j_1}, x_{0,j_0})$

We could call the elements of the list the 'digits of m'.

Example: The powers of the ideal (p) in Z.

$M = \mathbb{Z}$, $M_i = (p^i)$. Coset reps

$\{x_{0,j}\} = \{0, \ldots, p-1\}$

$\{x_{i,j}\} = \{0p^i, 1p^i, \ldots, (p-1)p^i\}$.

Write $m \in \mathbb{Z}$ as

$m = a_i p^i + a_{i-1} p^{i-1} + \cdots + a_1 p + a_0$

$a_j \in \{0, \ldots, p-1\}$. List $(0, 0, a_i p^i, \ldots a_1 p, a_0)$

Proposition.
Elements m, m′ in M produce the same list of coset representatives if and only if
$m - m' \in \bigcap M_i$

Thus $\bigcap M_i = 0$ implies elements of M are determined by these lists of elements.

Proposition.
A sequence (m_k) of elements of M is a Cauchy sequence if and only if

$\forall \epsilon > 0 \ \exists N$ so that $i, j > N$

$\Rightarrow d(m_i, m_j) < \epsilon$

In digit notation this means

$\forall \epsilon > 0 \ \exists N, \ i, j > N \Rightarrow$ first $\frac{1}{\epsilon}$ digits of m_i, m_j are the same.

Proposition
A Cauchy sequence in M determines a single list of digits that coincides with arbitrarily long initial segments of the digits of terms in the sequence.
Two Cauchy sequences are equivalent if and only if they produce the same sequence of digits.

March 4

On Friday I will teach online.

Definition. The completion of M (with respect to the filtration) is

Usual situation.

Question.
Is there any reason in normal high-school arithmetic that we work with the real numbers (rather than the algebraic numbers, for instance).

Is it anything more than notational convenience?

A Yes

B No

Example.

$\subseteq$

Some properties.

(a)

(b)

Categorical approach

Definition.

The limit, or inverse limit of the diagram is

Proposition A sequence (m_k) of elements of M is a Cauchy sequence $\iff$

$\forall\, d\ \exists\, N$ so that $k > N$

$\Rightarrow$ the first d digits of m_k are the same list.

A Cauchy sequence determines a list of digits (~~so~~ the $\forall d\ \exists\, N,\ k > N \Rightarrow$ the) whose first d terms are the d digits mentioned above

Propn. Two cauchy sequences are equivalent $\iff$ they produce the same sequence of digits.

Definition The completion $\hat{M}$ of M wrt the filtration is the set of equivalence classes of Cauchy sequences.

It is an R-module

If $M = R$ and the M_i are ideals then $\hat{M}$ is a ring.

Usual situation $\mathfrak{m} \subseteq R$ is an ideal in a commutative ring.

$R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \ldots$

We get $\hat{R_{\mathfrak{m}}}$. I write $\mathbb{Z}_p$ instead of $\hat{\mathbb{Z}_{(p)}}$. $\mathbb{Z}_{(p)}$ is the localiza

Example. $\mathfrak{m} = (t) \subseteq R[t]$.

Then $R[t]^{\wedge}_{\mathfrak{m}}$

$= R[[t]]$

$= \left\{ \begin{array}{l}\text{'formal'} \\ \text{power series}\end{array} \sum_{i \geq 0} a_i t^i \right\}$.

Properties

- $M^{\wedge}_{\mathfrak{m}} \cong R^{\wedge}_{\mathfrak{m}} \otimes_R M$

- $M^{\wedge}$ acquires a distance function extending d

- there is a homomorphism

$$M \longrightarrow M^{\wedge}$$
$$m \longmapsto \text{Cauchy sequence } (m, m, m, \dots)$$

with kernel $\bigcap_{i \geq 0} M_i$.

Categorical approach.

A diagram of R-modules is a functor $D: \mathcal{C} \to R\text{-mod}$ for some category $\mathcal{C}$.

e.g. $\mathcal{C} =$

$$= F(\cdots 4 \to 3 \to 2 \to 1 \to 0)$$

is the free category on the above quiver.

A diagram of shape $\mathcal{C}$ is a list of maps of R-modules

$$U_2 \to U_1 \to U_0$$

From a chain of submodules $M_{i+1} \hookrightarrow M_i$.

form the chain

$$\cdots \to M/M_2 \to M/M_1 \to M/M_0 = 0$$

The limit = inverse limit of D

$\varprojlim D$ is an R-module with maps a diagram so that

$$\begin{tikzcd} \varprojlim D \arrow[d, "\pi(x)"'] \arrow[dr, "\pi(y)"] & \\ F(x) \arrow[r, "F(\alpha)"] & F(y) \end{tikzcd}$$

all Δs commute and universal among such

The colimit = direct limit is ...

In R-mod limits always exist

$$\varprojlim D \subseteq \prod_{x \in Ob(\mathcal{C})} F(x)$$

where $(\ldots u_x, u_y \ldots)$ has $u_y = F(\alpha) u_x$ $\forall\, \alpha : x \to y$ in $\mathcal{C}$.