Example: the p-adic integers (Section 7.1)

The 10-adic integers

Decimal numbers are strings

finite integers | $a_3 a_2 a_1 a_0 \bullet a_{-1} a_{-2}$ - - - - - -

~~? • 0 0 0~~

10-adic integers are strings

$\cdots - a_3 a_2 a_1 a_0 \bullet 000$

Example

$$
\begin{array}{r}
\cdots 54321 \\
+ \cdots 41798 \\
\hline
\cdots 96119
\end{array}
$$

Let $p = 2$ and work to base 2.

This was wrong

$$
\begin{array}{r}
1\,?0\,?1\,?1\,?0\,?1. \\
\times \qquad 5. \\
\hline
0\,0\,0\,1. \\
1\,1\,1\,1\,1\,1\,1\,0
\end{array}
$$

5 is a notation for 101

$5\,(\ldots 01101101) = 1$ ✗

001

$\frac{1}{5} = \ldots \overline{0110}1101.$ ✗

It should be

$\frac{1}{5} = \ldots \overline{1100}11001101.$

Pre-class Warm-up!!

In the 2-adic integers, calculate

…1111.000 + 1.000 $= 0$

A 0

$-1 = \ldots 1111.$

B 1

C -1

D 10

E 11

Fact: The 2-adic integers are a group under +

Calculate also …$\overline{10}$101011.000 x 3
(where 3 is a notation for the 2-adic integer 11.000)

$$
\begin{array}{rr}
 & \ldots 01010111 \\
\times & 11 \\
\hline
 & \ldots 10101011 \\
+ & 010101110 \\
\hline
 & \ldots 000001
\end{array}
$$

First definition of completion

Let M be an R-module and consider a filtration

$$\cdots \cdots M_i \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

filtration = chain of submodules.

We define a pseudo-distance function on M.
Let $0 < \mu < 1$. We put, for $a, b \in M$

$$d(a,b) = \begin{cases} \mu^i & \text{if } a-b \in M_i - M_{i+1} \\ 0 & \text{if } a-b \in \bigcap_{i \geq 0} M_i \end{cases}$$

Proof of ultrametic $\leq$

If $a-b \in M_i - M_{i+1}$, $b-c \in M_j - M_{j+1}$ with $\mu^i \geq \mu^j$ then $i \leq j$ so $b-c \in M_i$, $a-c = a-b+b-c \in M_i$ so $d(a,c) \leq \mu^i$. $\square$

Proposition For all a, b in M we have

$d(a,b) \geq 0$, $d(a,b) = d(b,a)$

$$d(a,c) \leq \max(d(a,b), d(b,c)) \quad \text{the ultrametric inequality}$$

$$\leq d(a,b) + d(b,c)$$

$$d(a,b) = 0 \iff a-b \in \bigcap_{i \geq 0} M_i$$

Definition (Krull topology) We put a topology on M with basic open sets the balls $B_\epsilon(a)$, $a \in M$, $\epsilon > 0$.

Open sets = unions of these balls.

Proposition / Exercise. This topology is Hausdorff if and only if $\bigcap M_i = \{0\}$

The following is a collection of basic open sets: $a + M_i$, $i = 0, 1, 2, \ldots$ $a \in M$.

$\quad = B_{\mu^{i-1}}(a)$

M is a topological group under +

Representing elements of M as sequences

Take a set of coset representatives

$x_{i,j}$ for M_{i+1} in M_i, j varies $j = 1, 2, 3, \ldots$

Each $m \in M$ determines a list of these coset reps: Consider $i = 0$

$M_0 \supseteq M_1$. $m \equiv x_{0,j_0} \pmod{M_1}$

$m - x_{0,j_0} \equiv x_{1,j_1} \pmod{M_2}$

$m - x_{0,j_0} - x_{1,j_1} \equiv x_{2,j_2} \pmod{M_3}$

We get a list $(\ldots, x_{2,j_2}, x_{1,j_1}, x_{0,j_0})$

We could call the elements of the list the 'digits of m'.

Example: The powers of the ideal (p) in $\mathbb{Z}$.

$M = \mathbb{Z}$, $M_i = (p^i)$. Coset reps

$\{x_{0,j}\} = \{0, \ldots, p-1\}$

$\{x_{i,j}\} = \{0p^i, 1p^i, \ldots, (p-1)p^i\}$.

Write $m \in \mathbb{Z}$ as

$m = a_i p^i + a_{i-1} p^{i-1} + \cdots + a_1 p + a_0$

$a_j \in \{0, \ldots, p-1\}$. List $(0, 0, a_i p^i, \ldots a_1 p, a_0)$

Proposition.
Elements m, m' in M produce the same list of coset representatives if and only if $m - m' \in \bigcap M_i$

Thus $\bigcap M_i = 0$ implies elements of M are determined by these lists of elements.

Pre-class Warm-up!!

Given Cauchy sequences (a_i) and (b_i) in a metric space, in defining the completion of the space, which (if any) of the following means that the Cauchy sequences are equivalent?

A There exists a point x in the space so that $a_i \to x$ and $b_i \to x$ as $i \to \infty$.

B Given $\epsilon > 0$ $\exists N$ so that $i > N$ implies $d(a_i, b_i) < \epsilon$

C Given $\epsilon > 0$ $\exists N$ so that $i, j > N$ implies $d(a_i, b_j) < \epsilon$

D Given N, $\exists \epsilon > 0$ so that $i > N$ $\Rightarrow d(a_i, b_i) < \epsilon$

E None of the above.

Example: $(3, 3.1, 3.14, 3.141, 3.1415, \ldots)$ In $\mathbb{Q}$ is a Cauchy sequence with no limit.

Defn. A sequence (a_i) is Cauchy $\Leftrightarrow$ $\forall \epsilon > 0$ $\exists N$ so that $i, j > N$ $\Rightarrow d(a_i, a_j) < \epsilon$

Sequences (a_i) (b_i) are equivalent $\Leftrightarrow$ B or C opposite.

Defn. The completion $\hat{M}$ of the metric space M is the set of equivalence classes of Cauchy sequences.

We have an embedding (metric d, not pseudometric) $M \longrightarrow \hat{M}$

$m \longmapsto (m, m, m, m, \ldots)$

$\hat{M}$ acquires a metric $\hat{d}$ extending d.

$\hat{M}$ is complete.

2nd question: For what reason did we ever learn about Cauchy sequences in the first place?

Proposition.
A sequence (m_k) of elements of M is a Cauchy sequence if and only if

$\forall \epsilon > 0 \ \exists N$ so that $i, j > N$
$\Rightarrow d(m_i, m_j) < \epsilon$

In digit notation this means
$\forall \epsilon > 0 \ \exists N,\ i, j > N \Rightarrow$
first $\frac{1}{\epsilon}$ digits of m_i, m_j are the same.

Proposition
A Cauchy sequence in M determines a single list of digits that coincides with arbitrarily long initial segments of the digits of terms in the sequence.
Two Cauchy sequences are equivalent if and only if they produce the same sequence of digits.

$d(m, m') < \mu^i \Leftrightarrow$ the first i digits of m, m' are the same.

Definition. The completion $\hat{M}$ of M (with respect to the filtration) is the set of equivalence classes of Cauchy sequences in M.

Usual situation. We take an ideal $\mathfrak{m} \subseteq R$
$M = R,\ M_i = \mathfrak{m}^i$

$, R = \mathbb{Z} = M,\ M_i = (2^i)$

Question. Let M_i = (2^i)⊊ Z and let μ = 1/2.
What is d(48,96)?

$96 - 48 = 48 \in (2^4) - (2^5)$ so $i = 4$
$d(48, 96) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

Proposition. The completion is
a. an R-module,
b. a metric space
c. the same as the completion of $M/\bigcap_{i\geq 0} M_i$
d. if M = R and the M_i are ideals, then it is a ring.

Proof a. We can add Cauchy sequences, adding an equivalent sequence produces equivalent sequences

b. done

c. If $m_i \in \bigcap_{i\geq 0} M_i$ then

$a_1+m_1, a_2+m_2, \ldots$

is equivalent to (a_i).

Every sequence (a_i') with a_i' in the same coset of $\bigcap M_i$ as a_i is in the same equivalence class as (a_i)

the set of sequences

$(\ldots\ldots, x_{2,j_2}, x_{1,j_1}, x_{0,j_0})$

with $x_{i,j_i} \in$ coset reps of M_{i+1} in M_i

because every equiv. class of Cauchy sequences produces such a list of coset reps and is determined by this list.

Given such a list we construct a Cauchy sequence

$x_{0,j_0}, x_{0,j_0}+x_{1,j_1}, x_{0,j_0}+x_{1,j_1}+x_{2,j_2}, \ldots$

Quesion: Is it obvious that the completion $\hat{M}$ is an R-module?

A Yes, it is obvious.

B No, it is not obvious.

Question.
Is there any reason in normal high-school arithmetic that we work with the real numbers (rather than the algebraic numbers, for instance).

Is it anything more than notational convenience?

A Yes

B No

Example.

$W \quad \subseteq$

Some more properties.

(a)

(b) there is a homomorphism

Categorical approach

Definition.
A diagram of R-modules

The limit, or inverse limit of the diagram is

Example. $\mathfrak{m} = (t) \subseteq R[t]$.

Then $R[t]^{\wedge}_{\mathfrak{m}}$

$= R[[t]]$

$= \left\{ \begin{array}{l}\text{'formal'} \\ \text{power series}\end{array} \sum_{i \geq 0} a_i t^i \right\}$.

Properties

- $M^{\wedge}_{\mathfrak{m}} \cong R^{\wedge}_{\mathfrak{m}} \otimes_R M$
- $M^{\wedge}$ acquires a distance function extending d
- there is a homomorphism

$$M \longrightarrow M^{\wedge}$$
$$m \longmapsto \text{Cauchy sequence } (m, m, m, \dots)$$

with kernel $\bigcap_{i \geq 0} M_i$.

Categorical approach.

A diagram of R-modules is a functor $D: \mathcal{C} \to R\text{-mod}$ for some category $\mathcal{C}$.

e.g. $\mathcal{C} =$

$= F(\cdots 4 \to 3 \to 2 \to 1 \to 0)$

is the free category on the above quiver.

A diagram of shape $\mathcal{C}$ is a list of maps of R-modules

$$U_2 \to U_1 \to U_0$$

From a chain of submodules $M_{i+1} \hookrightarrow M_i$.

form the chain

$$\cdots \to M/M_2 \to M/M_1 \to M/M_0 = 0$$

The limit = inverse limit of D

$\varprojlim D$ is an R-module with maps a diagram so that

$$\begin{tikzcd} \varprojlim D \arrow[d, "\pi(x)"'] \arrow[dr, "\pi(y)"] & \\ F(x) \arrow[r, "F(\alpha)"] & F(y) \end{tikzcd}$$

all Δs commute and universal among suc...

The colimit = direct limit is ...

In R-mod limits always exist

$$\varprojlim D \subseteq \prod_{x \in Ob(\mathcal{C})} F(x)$$

where $(\ldots u_x, u_y \ldots)$ has $u_y = F(\alpha) u_x$ $\forall\ \alpha: x \to y$ in $\mathcal{C}$.