

## Chapter 7: Completions

Example: the p-adic integers (Section 7.1)

The 10-adic integers

Decimal numbers are strings

$\begin{array}{l} \text{finite} \\ \text{integers} \end{array} \left| \begin{array}{l} a_3 a_2 a_1 a_0 \cdot a_{-1} a_{-2} \dots \\ \text{---} \end{array} \right.$

~~integers~~

10-adic integers are strings

...  $- a_3 a_2 a_1 a_0 \cdot 000$

Example

$$\begin{array}{r} \dots 54321 \\ + \dots 41798 \\ \hline - \dots 96119 \\ \quad \quad \quad 11 \end{array}$$

Let  $p = 2$  and work to base 2.

*This was wrong*

$(1) \ ? \ ? \ ? \ ? \ 0 \ ? \ 1 \cdot$   
5.

$\times$

---

1 1 1 1 0 0 0 1.

5 is a notation for 101

5  $(\dots 01101101) = 1 \times$

001

$\frac{1}{5} = \dots \overline{01101101} \cdot \times$

*It should be*

$\frac{1}{5} = \dots \overline{110011001101} \cdot$

# Pre-class Warm-up!!

In the 2-adic integers, calculate

$$\dots 1111.000 + 1.000 = \bigcirc$$

A  $0 \quad -1 = \dots 1111.$

B  $1$

C  $-1$

D  $10$

E  $11$

Fact: The 2-adic integers are a group under  $+$

Calculate also  $\dots \overline{10101011.000} \times 3$   
(where 3 is a notation for the 2-adic integer 11.000)

$$\begin{array}{r} \dots 0101011 \\ \times \quad \quad \quad \quad \quad 11 \\ \hline \dots 10101011 \\ + \quad 01010110 \\ \hline \dots 000001 \\ \quad \quad \quad | \quad | \quad | \quad | \end{array}$$

## First definition of completion

Let  $M$  be an  $R$ -module and consider a filtration

$$\dots \subseteq M_i \subseteq \dots \subseteq M_1 \subseteq M_0 = M$$

filtration = chain of submodules.  
pseudo-

We define a distance function on  $M$ .

Let  $0 < \mu < 1$ . We put, for  $a, b \in M$

$$d(a, b) = \begin{cases} \mu^i & \text{if } a-b \in M_i - M_{i+1} \\ 0 & \text{if } a-b \in \bigcap_{i \geq 0} M_i \end{cases}$$

Proof of ultrametric  $\leq$

If  $a-b \in M_i - M_{i+1}$ ,  $b-c \in M_j - M_{j+1}$

with  $\mu^i \geq \mu^j$  then  $i \leq j$  so

$b-c \in M_i$ ,  $a-c = a-b + b-c \in M_i$

so  $d(a, c) \leq \mu^i$ .  $\square$

Proposition For all  $a, b$  in  $M$  we have

$$d(a, b) \geq 0, \quad d(a, b) = d(b, a)$$

$$d(a, c) \leq \max(d(a, b), d(b, c)) \quad \text{the ultrametric inequality}$$
$$\leq d(a, b) + d(b, c)$$

$$d(a, b) = 0 \Leftrightarrow a-b \in \bigcap_{i \geq 0} M_i$$

Definition (Krull topology) We put a topology on  $M$  with basic open sets the balls

$$B_\epsilon(a), \quad a \in M, \quad \epsilon > 0.$$

Open sets = unions of these balls.

Proposition / Exercise. This topology is Hausdorff if and only if  $\bigcap M_i = \{0\}$

The following is a collection of basic open sets:  $a + M_i$ ,  $i = 0, 1, 2, \dots$   $a \in M$ .  
"  $B_{\mu^{i-1}}(a)$

$M$  is a topological group under  $+$

## Representing elements of $M$ as sequences

Take a set of coset representatives

$x_{i,j}$  for  $M_{i+1}$  in  $M_i$ ,  $j$  varies

Each  $m \in M$  determines a list of these coset reps: Consider  $i=0$

$$M_0 \supseteq M_1 \dots m \equiv x_{0,j_0} \pmod{M_1}$$

$$m - x_{0,j_0} \equiv x_{1,j_1} \pmod{M_2}$$

$$m - x_{0,j_0} - x_{1,j_1} \equiv x_{2,j_2} \pmod{M_3}$$

We get a list  $(\dots, x_{2,j_2}, x_{1,j_1}, x_{0,j_0})$

We could call the elements of the list the 'digits of  $m$ '.

Example: The powers of the ideal  $(p)$  in  $\mathbb{Z}$ .

$M = \mathbb{Z}$ ,  $M_i = (p^i)$  Coset reps

$$\{x_{0,j}\} = \{0, \dots, p-1\}$$

$$\{x_{i,j}\} = \{0p^i, 1p^i, \dots, (p-1)p^i\}$$

Write  $m \in \mathbb{Z}$  as

$$m = a_i p^i + a_{i-1} p^{i-1} + \dots + a_1 p + a_0$$

$a_j \in \{0, \dots, p-1\}$ . List  $(0, a_i p^i, \dots, a_1 p, a_0)$

Proposition.

Elements  $m, m'$  in  $M$  produce the same list of coset representatives if and only if  $m - m' \in \bigcap M_i$

Thus  $\bigcap M_i = 0$  implies elements of  $M$  are determined by these lists of elements.

# Pre-class Warm-up!!

Given Cauchy sequences  $(a_i)$  and  $(b_i)$  in a metric space, in defining the completion of the space, which (if any) of the following means that the Cauchy sequences are equivalent?

- A There exists a point  $x$  in the space so that  $a_i \rightarrow x$  and  $b_i \rightarrow x$  as  $i \rightarrow \infty$ .
- B Given  $\epsilon > 0 \exists N$  so that  $i > N$  implies  $d(a_i, b_i) < \epsilon$
- C Given  $\epsilon > 0 \exists N$  so that  $i, j > N$  implies  $d(a_i, b_j) < \epsilon$
- D Given  $N, \exists \epsilon > 0$  so that  $i > N \Rightarrow d(a_i, b_i) < \epsilon$
- E None of the above.

Example:  $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$   
In  $\mathbb{Q}$  is a Cauchy sequence with no limit.

Defn. A sequence  $(a_i)$  is Cauchy  
 $\Leftrightarrow \forall \epsilon > 0 \exists N$  so that  $i, j > N \Rightarrow d(a_i, a_j) < \epsilon$

Sequences  $(a_i)$   $(b_i)$  are equivalent  
 $\Leftrightarrow B$  or  $C$  opposite.

Defn. The completion of the metric space  $M$  is the set of equivalence classes of Cauchy sequences.

We have an embedding (metric, not pseudometric)  $M \rightarrow \hat{M}$

$m \mapsto (m, m, m, m, \dots)$   
 $\hat{M}$  acquires a metric  $\hat{d}$  extending  $d$ .  
 $M$  is complete.

2nd question: For what reason did we ever learn about Cauchy sequences in the first place?

Proposition.

A sequence  $(m_k)$  of elements of  $M$  is a Cauchy sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ so that } i, j > N$$

$$\Rightarrow d(m_i, m_j) < \epsilon$$

In digit notation this means

$$\forall \epsilon > 0 \exists N, i, j > N \Rightarrow$$

first  $\frac{1}{\epsilon}$  digits of  $m_i, m_j$  are the same.

Proposition

A Cauchy sequence in  $M$  determines a single list of digits that coincides with arbitrarily long initial segments of the digits of terms in the sequence.

Two Cauchy sequences are equivalent if and only if they produce the same sequence of digits.

$$d(m, m') < \mu^i \Leftrightarrow \text{the first } i \text{ digits of } m, m' \text{ are the same.}$$

Definition. The completion  $\hat{M}$  of  $M$  (with respect to the filtration) is the set of equivalence classes of Cauchy sequences in  $M$ .

Usual situation. We take an ideal  $M \subseteq R$   
 $M = R, M_i = M^i$

$$R = \mathbb{Z} = M, M_i = (2^i)$$

Question. Let  $M_i = (2^i) \subseteq \mathbb{Z}$  and let  $\mu = 1/2$ .  
What is  $d(48, 96)$ ?

$$96 - 48 = 48 \in (2^4) - (2^5) \text{ so } i = 4$$
$$d(48, 96) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

Proposition. The completion is

- a. an R-module,
- b. a metric space
- c. the same as the completion of  $M / \bigcap_{i \geq 0} M_i$
- d. if  $M = R$  and the  $M_i$  are ideals, then it is a ring.

Proof a. We can add Cauchy sequences, adding an equivalent sequence produces equivalent sequences

b. done

c. If  $m_i \in \bigcap_{i \geq 0} M_i$  then

$$a_1 + m_1, a_2 + m_2, \dots$$

is equivalent to  $(a_i)$ .

Every sequence  $(a'_i)$  with

$a'_i$  in the same coset of  $\bigcap M_i$

as  $a_i$  is in the same equivalence class as  $(a_i)$

the set of sequences  
 $(\dots, x_{2,j_2}, x_{1,j_1}, x_{0,j_0})$   
with  $x_{i,j_i} \in$  set reps of  $M_{i+1}$  in  $M_i$

because every equiv. class of Cauchy sequences produces such a list of set reps and is determined by this list.

Given such a list we construct a Cauchy sequence

$$x_{0,j_0}, x_{0,j_0} + x_{1,j_1}, x_{0,j_0} + x_{1,j_1} + x_{2,j_2}, \dots$$

Question: Is it obvious that the completion  $\hat{M}$  is an R-module?

A Yes, it is obvious.

B No, it is not obvious.



Question.

Is there any reason in normal high-school arithmetic that we work with the real numbers (rather than the algebraic numbers, for instance).

Is it anything more than notational convenience?

A Yes

B No

# Pre-class Warm-up!!

For each prime  $p$  we have constructed the  $p$ -adic integers.

Are the  $p$ -adic integers

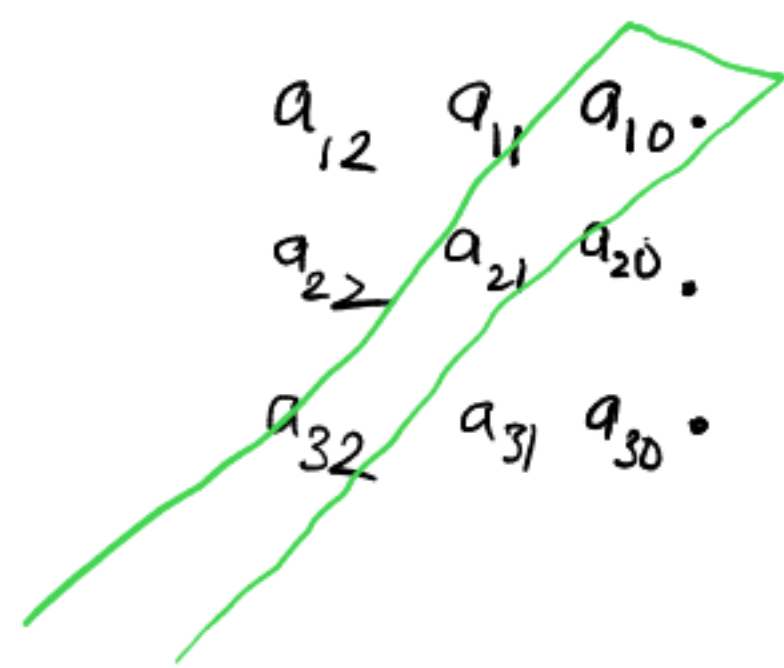
countable?

✓

$p$ -adic integers are lists

$\dots a_2 a_1 a_0 \bullet$

If we could enumerate such lists, make a table



Get a list different from the diagonal list in each place.

This expansion is not in the enumeration.

Notation and example. If  $\mathcal{M}$  is an ideal of  $R$ ,  $M$  is an  $R$ -module we get a filtration

$$\dots \subseteq \mathcal{M}^2 M \subseteq \mathcal{M} M \subseteq M$$

The limit is denoted  $\hat{M}_{\mathcal{M}} = M_{\mathcal{M}}^{\wedge}$ . In particular  $R_{\mathcal{M}}^{\wedge}$  or  $\hat{R}_{\mathcal{M}}$  is the completion of  $R$  at  $\mathcal{M}$ .

$$R_{\mathcal{M}}^{\wedge} =: R[[t]] = \left\{ \sum_{i \geq 0} a_i t^i \mid a_i \in R \right\}$$

Get this description by taking coset reps  $a t^i$ ,  $a \in R$  for  $(t)^{i+1}$  in  $(t)^i$ .

Some more properties we have already seen:

Proposition

a. There is a homomorphism  $M \rightarrow \hat{M}$  with kernel  $m \mapsto (m, m, m, \dots)$

b. The distance function on  $M_{\mathcal{M}}$  extends to a distance function on  $\hat{M}$  and  $\hat{M}$  is complete.

Proposition. Let  $\mathcal{M}$  be an ideal of  $R$ . Then the ideal  $(\mathcal{M})$  of  $R_{\mathcal{M}}^{\wedge}$  generated by  $\mathcal{M}$  has

$$R_{\mathcal{M}}^{\wedge} / (\mathcal{M}) \cong R / \mathcal{M},$$

$$\text{and } (\mathcal{M}) \cap R = \mathcal{M}$$

$$\text{Example: } \mathbb{Z}_{(p)}^{\wedge} / (p) \cong \mathbb{Z} / p\mathbb{Z}$$

Proof Elements of  $\mathcal{M}$  are represented by lists of coset reps  $(x_1, x_2, \dots, 0)$

On completing,  $\mathcal{M}$  completes to all lists  $(?, ?, ?, 0)$ . This is the ideal of  $R_{\mathcal{M}}^{\wedge}$  generated by  $\mathcal{M}$ .

The lists  $(0, 0, \dots, 0, a)$ ,  $a$  is a coset rep for  $\mathcal{M}$  in  $R$  are coset reps for  $(\mathcal{M})$  in  $R_{\mathcal{M}}^{\wedge}$ .

This means  $R / \mathcal{M} \cong R_{\mathcal{M}}^{\wedge} / (\mathcal{M})$  are in bijection.  $(\mathcal{M}) \cap R = \mathcal{M}$  = lists of elements of  $R$  belonging to  $\mathcal{M}$ .  $R / \mathcal{M} \rightarrow R_{\mathcal{M}}^{\wedge} / (\mathcal{M})$  is an isomorphism.

## Categorical approach

Definition.

A diagram of  $R$ -modules is a functor

$F: \mathcal{C} \rightarrow R\text{-mod}$ , where  $\mathcal{C}$  is some category

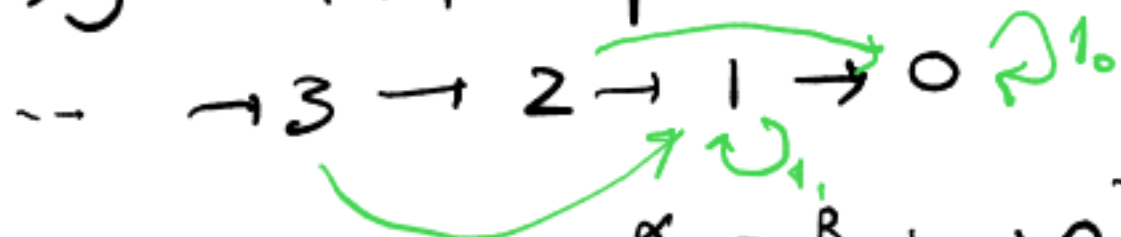
This is the same as a representation of  $\mathcal{C}$  over  $R$ .

Let  $\mathcal{C}$  be the category

with objects  $\{0, 1, 2, \dots\} = \mathbb{N}$

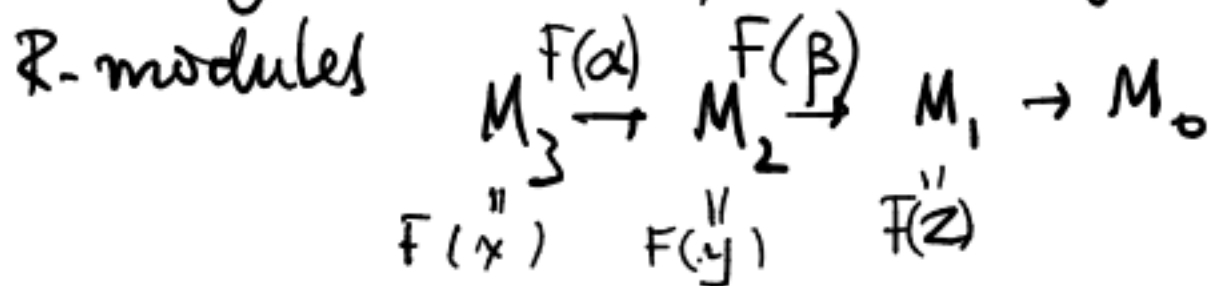
and a unique morphism  $i \rightarrow j$  whenever

$i \geq j$ . Picture of  $\mathcal{C}$ :

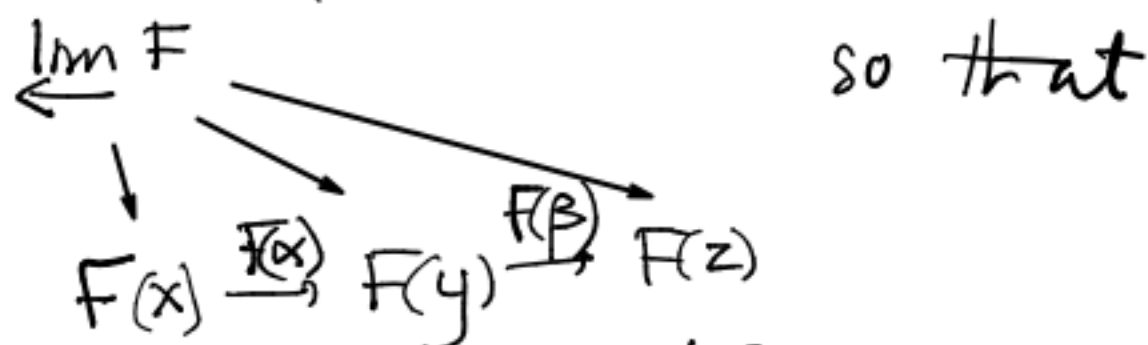


Free (quiver  $\dots \rightarrow 3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1 \rightarrow 0$ ).

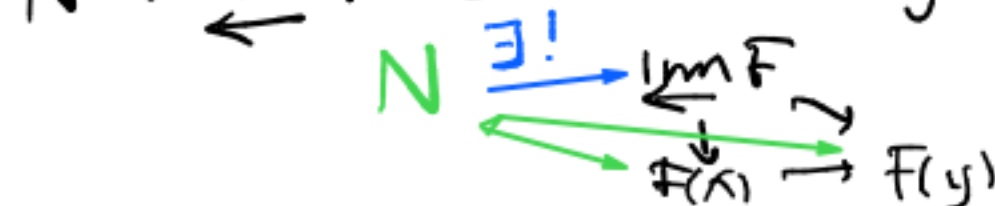
A diagram with shape  $\mathcal{C}$  is diagram of



The limit, or inverse limit of the diagram is a diagram of  $R$ -modules



- (1) All triangles commute
- (2) Given another such picture with  $N$  instead of  $\lim F$ ,  $\exists$  unique  $R$ -module hom  $N \rightarrow \lim F$  so that everything commutes.



Example. The pullback, the product.

Given  $\mathcal{C} = \bullet \rightarrow \bullet \leftarrow \bullet$  a diagram is a picture of  $R$ -modules.

$$\begin{array}{ccc}
 P & \rightarrow & N \\
 \downarrow & \text{comm} & \downarrow \\
 M & \rightarrow & L
 \end{array}$$

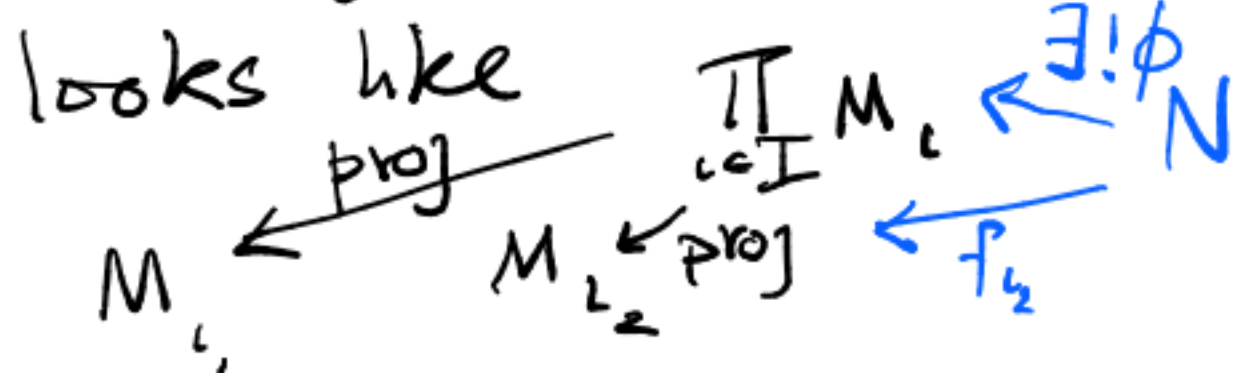
The pullback is a module  $P$  with morphisms as shown, so that (2) holds. The pullback is a limit.

The colimit, or direct limit of the diagram is similar, with arrows in the opposite direction.

The product of modules  $M_i, i \in I$ .

Take a category  $\mathcal{C}$  where the objects are the  $i$  in  $I$ , the only morphisms are identities.

A diagram  $F: \mathcal{C} \rightarrow R\text{-mod}$



Their product is the inverse limit of this diagram.

$$\prod M_i = (a_i) \quad a_i \in M_i.$$

$$\phi(n) = (f_i(n))$$

Proposition. Limits of  $R$ -modules always exist and are unique.

Corollary. Given a filtration of a module  $M$

the completion of  $M$  is the same thing as the inverse limit of the diagram

## Limits in terms of the constant diagram

Definition. Given an  $R$ -module  $N$  and a category  $C$ , the constant functor (or diagram)

Proposition. Given a diagram  $F : C \rightarrow R\text{-mod}$ , we have

The functor  $R\text{-mod} \rightarrow \text{Fun}(C, R\text{-mod})$  given by