Spectral sequences

Source: I prefer the treatment in
K.S. Brown, Cohomology of groups,
chapter VII

Topics:
- the spectral sequence of a filtered complex
- how these arise from double complexes
- application to the homology of a union of spaces.

Motivation

We know that a short exact sequence of chain complexes 0 -> A. -> B. -> C. -> 0 gives rise to a long exact sequence in homology, perhaps giving information about H_*(B.)

Examples

1. Ext groups

Given a s.e.s. of R-modules $0 \to L \to M \to N \to 0$ we get a s.e.s of chain complexes

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0$$

where $P \to A \to 0$ is a proj. res'n of A, hence a long e.s.

$$0 \to \mathrm{Ext}^0_R(A, L) \to \qquad \to$$

$\to$ . . . etc.

2. We may have a simplicial complex $X \cup Y$ where $X \cap Y$ is a subsimplicial complex

We have a s.e.s. of chain cxes

$$0 \to C_\cdot(X \cap Y) \to C_\cdot(X) \oplus C_\cdot(Y) \to C_\cdot(X \cup Y) \to 0$$

Get long e.s. in homology.

What if the simplicial complex Δ has several subcomplexes $X_1, \dots, X_n$.

$\Delta = \bigcup X_i$

$C_\cdot(X_i) \subseteq C_\cdot(\Delta)$.

Let $F_p(\Delta)$ = span of the simplices in Δ that lie in at least p of the $X_1, \dots, X_n$.

We get subcomplexes

$$\cdots F_3(\Delta) \subseteq F_2(\Delta) \subseteq F_1(\Delta) \subseteq F_0(\Delta) = C_\cdot(\Delta)$$

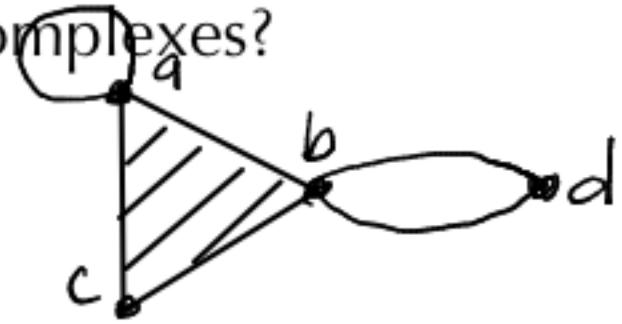
Can we get info about $H_*(C_\cdot(\Delta))$ from the $H_*(F_p(\Delta)/F_{p+1}(\Delta))$?

Yes!

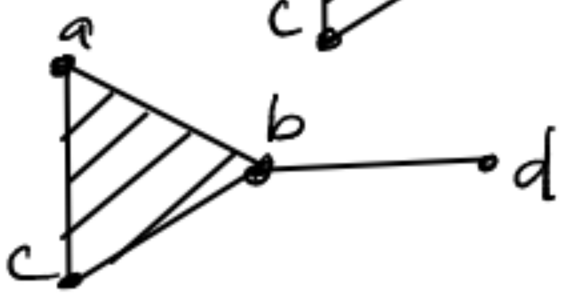
There is a spectral sequence generalizing the Mayer–Vietoris long e. s.

Pre-class Warm-up!!

Which of the following define the same simplicial complexes?

1. 

Not a simplicial complex

2. 

3. $\{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}\}$

A 1 and 2 describe the same simplicial complex.

B 1 and 3 describe the same simplicial complex.

C 2 and 3 describe the same simplicial complex.

D They all describe the same simplicial complex.

An (abstract) simplicial complex is a set Δ of subsets of a set S so that $T \in \Delta$, $U \subseteq T$ $\Rightarrow$ $U \in \Delta$.

Filtrations of modules and associated graded modules

An ascending filtration of a module M is a chain of submodules

$$\cdots \subseteq F_p(M) \subseteq F_{p+1} \subseteq \cdots \subseteq M.$$

A $\mathbb{Z}$ graded module is a list of modules M_p, $p \in \mathbb{Z}$.

We may want to think of it as $\bigoplus_{p \in \mathbb{Z}} M_p$.

Given a filtration the associated graded module $\mathrm{Gr}\, M$ has

$$\mathrm{Gr}_p M = F_p M / F_{p-1} M.$$

e.g. $k[x] = \bigoplus_{p \geq 0} k x^p$

We assume that filtrations are finite.

This means $F_p = F_{p+1} = \cdots$ if p is large enough and $F_p = F_{p-1} = \cdots$ if p is small enough.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

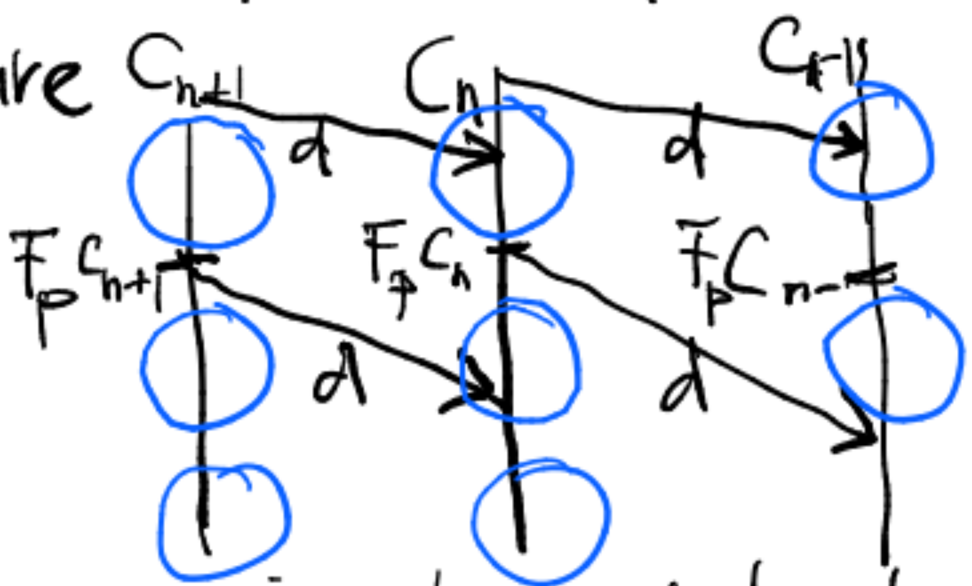
D Shaky

Definition.

A filtration of a chain complex C. is a ~~chain~~ list of subcomplexes

$$\cdots \subseteq F_p(C.) \subseteq F_{p+1}(C.) \subseteq \cdots$$

Picture



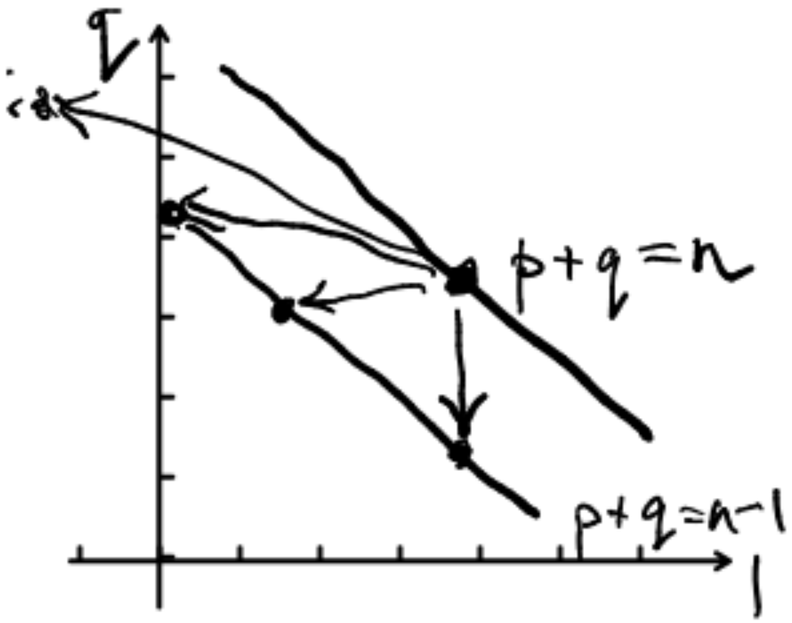
The associated graded object has

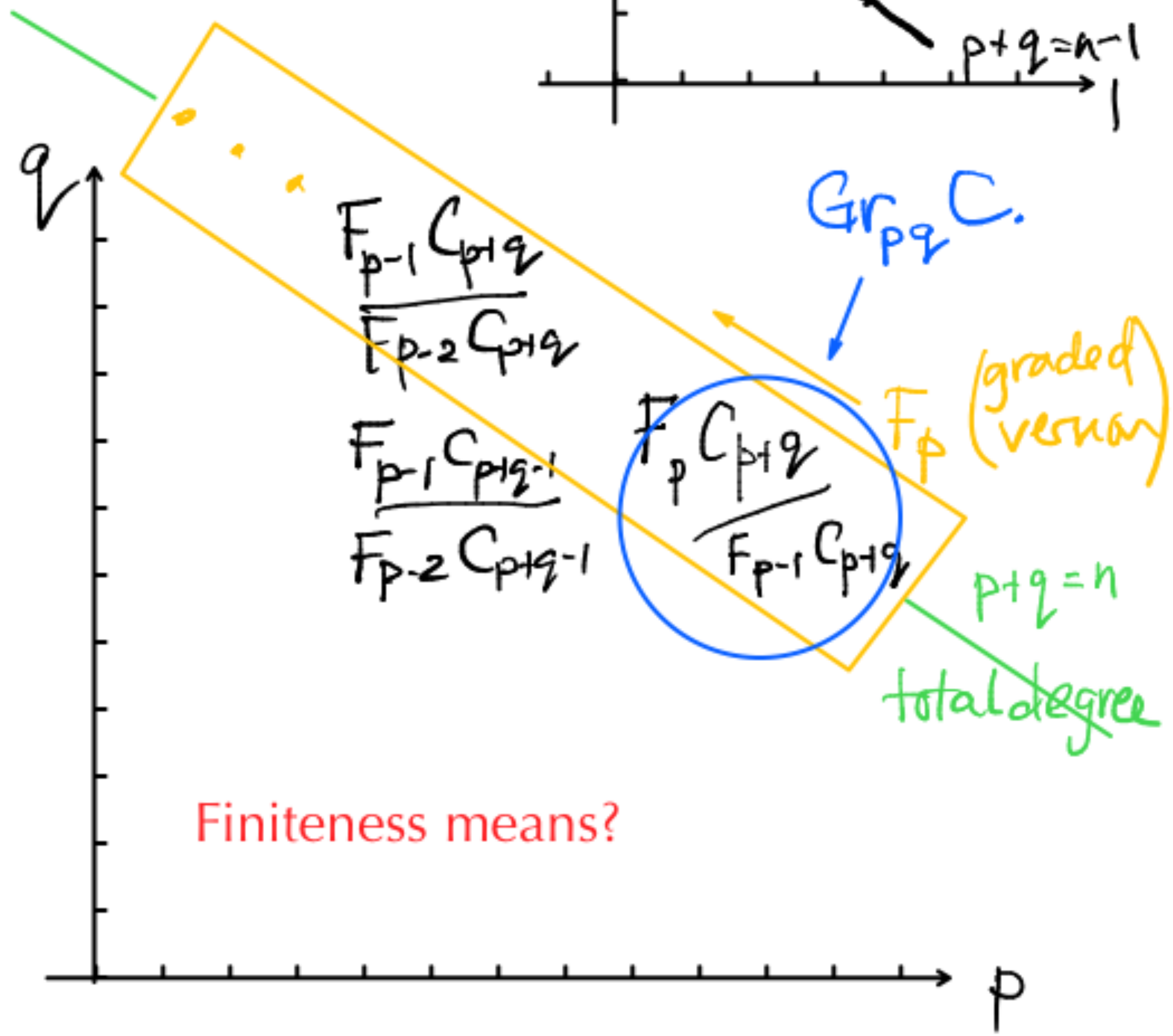
$$Gr_p C. = F_pC./F_{p-1}C.$$

which is also a list of modules indexed by homological degree n.

We consider filtrations finite in each homological degree.

The differential looks like





Finiteness means?

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The filtration on the homology of a filtered complex.

Proposition.

a. The image of H.(F_p C) in H.(C) is

$$(F_p C \cap Z)/(F_p C \cap B)$$

$$Gr_p H(C) \simeq (F_p C \cap Z)/((F_p C \cap B) + (F_{p-1} C \cap Z))$$

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The spectral sequence of a filtered complex

We define

$$Z^r_{pq} = F_p C_{p+q} \cap \partial^{-1} F_{p-r} C_{p+q-1}$$

$$Z^\infty_p = F_p C \cap Z$$

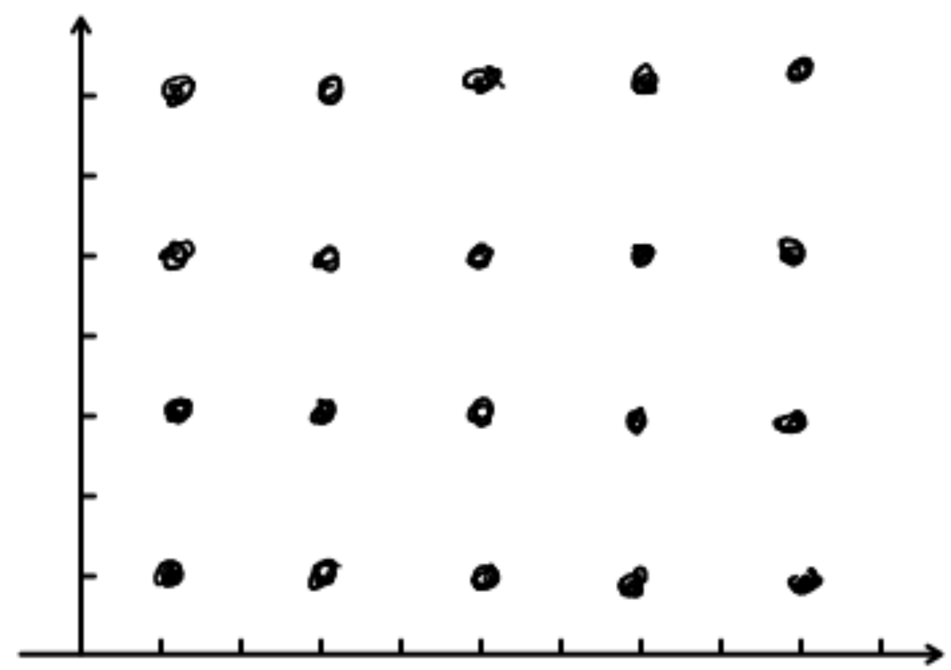
$$B^r_{pq} = F_p C_{p+q} \cap \partial F_{p+r-1} C_{p+q+1}$$

$$B^\infty_p = F_p C \cap B$$

Proposition.
Assume the filtration is finite in each homological degree. Then

$$B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^\infty_p \subseteq Z^\infty_p \subseteq \cdots$$
$$\subseteq Z^1_p \subseteq Z^0_p = F_p C$$

In each degree the B and Z sequences stabilize.



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Definition

$$E_p^r = Z_p^r / (B_p^r + Z_{p-1}^{r-1})$$

$$= Z_p^r / (B_p^r + (F_{p-1}C \cap Z_p^r))$$

$$E_1^{\infty} = Z_p^{\infty} / (B_p^{\infty} + Z_{p-1}^{\infty}) = Gr_p \, H(C)$$