Spectral sequences

Source: I prefer the treatment in
K.S. Brown, Cohomology of groups,
chapter VII

Topics:
- the spectral sequence of a filtered complex
- how these arise from double complexes
- application to the homology of a union of spaces.

Motivation

We know that a short exact sequence of chain complexes 0 -> A. -> B. -> C. -> 0 gives rise to a long exact sequence in homology, perhaps giving information about H_*(B.)

Examples

1. Ext groups

Given a s.e.s. of R-modules $0 \to L \to M \to N \to 0$ we get a s.e.s of chain complexes

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N) \to 0$$

where $P \to A \to 0$ is a proj. res^{dn} of A, hence a long e.s.

$$0 \to \mathrm{Ext}^0_R(A, L) \to \quad \to \quad \ldots \text{etc.}$$

2. We may have a simplicial complex $X \cup Y$ where $X \cap Y$ is a subsimplicial complex

We have a s.e.s. of chain cxes

$$0 \to C_\cdot(X \cap Y) \to C_\cdot(X) \oplus C_\cdot(Y) \to C_\cdot(X \cup Y) \to 0$$

Get long e.s. in homology.

What if the simplicial complex Δ has several subcomplexes

$X_1, \ldots, X_n$.

$\Delta = \bigcup X_i$.

$C_\cdot(X_i) \subseteq C_\cdot(\Delta)$.

Let $F_p(\Delta)$ = span of the simplices in Δ that lie in at least p of the $X_1, \ldots, X_n$.

We get subcomplexes

$$\cdots F_3(\Delta) \subseteq F_2(\Delta) \subseteq F_1(\Delta) \subseteq F_0(\Delta) = C_\cdot(\Delta)$$

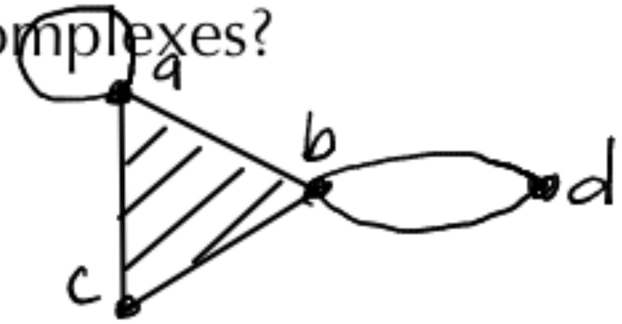
Can we get info about $H_*(C_\cdot(\Delta))$ from the $H_*\left(F_p(\Delta)/F_{p+1}(\Delta)\right)$?

Yes!

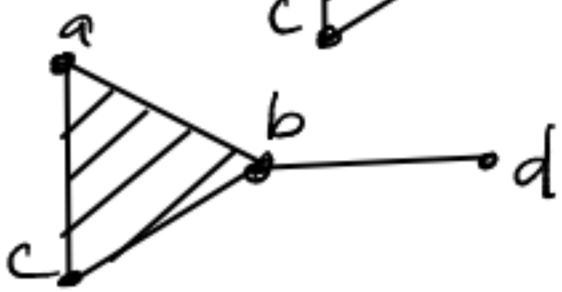
There is a spectral sequence generalizing the Mayer–Vietoris long e. s.

Pre-class Warm-up!!

Which of the following define the same simplicial complexes?

1. 

Not a simplicial complex

2. 

3. $\{\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{a,b,c\}\}$

A 1 and 2 describe the same simplicial complex.

B 1 and 3 describe the same simplicial complex.

C 2 and 3 describe the same simplicial complex.

D They all describe the same simplicial complex.

An (abstract) simplicial complex is a set Δ of subsets of a set S so that $T \in \Delta$, $U \subseteq T$ $\Rightarrow$ $U \in \Delta$.

Filtrations of modules and associated graded modules

An ascending filtration of a module M is a chain of submodules

$$\cdots \subseteq F_p(M) \subseteq F_{p+1} \subseteq \cdots \subseteq M.$$

A $\mathbb{Z}$-graded module is a list of modules M_p, $p \in \mathbb{Z}$.
We may want to think of it as $\bigoplus_{p \in \mathbb{Z}} M_p$.

Given a filtration the associated graded module $\mathrm{Gr}\, M$ has

$$\mathrm{Gr}_p M = F_p M / F_{p-1} M.$$

e.g. $k[x] = \bigoplus_{p \geq 0} k x^p$

We assume that filtrations are finite.

This means $F_p = F_{p+1} = \cdots$ if p is large enough. and $F_p = F_{p-1} = \cdots$ if p is small enough.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

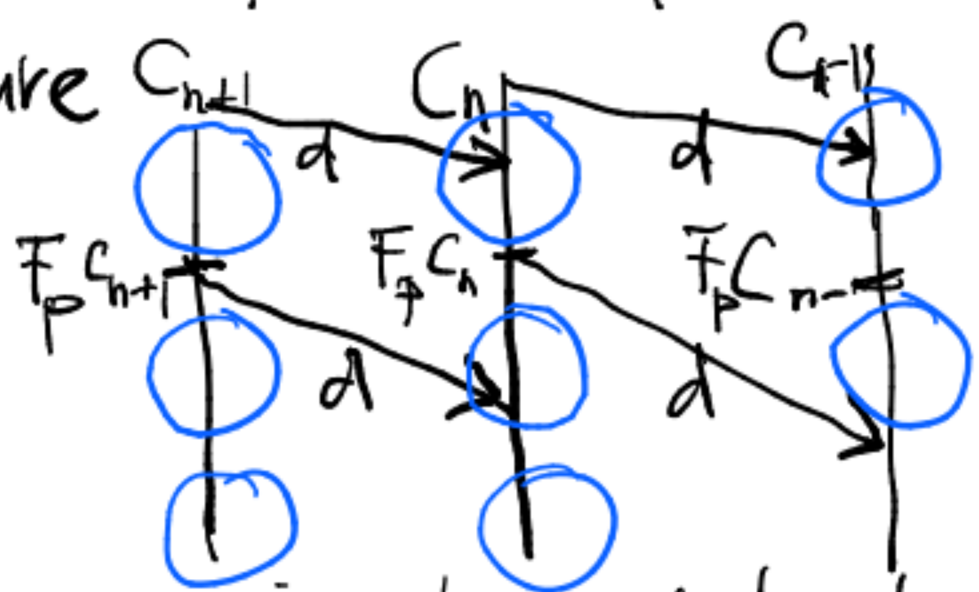
D Shaky

Definition.

A filtration of a chain complex C. is a chain (list) of subcomplexes

$$\cdots \subseteq F_p(C.) \subseteq F_{p+1}(C.) \subseteq \cdots$$

Picture

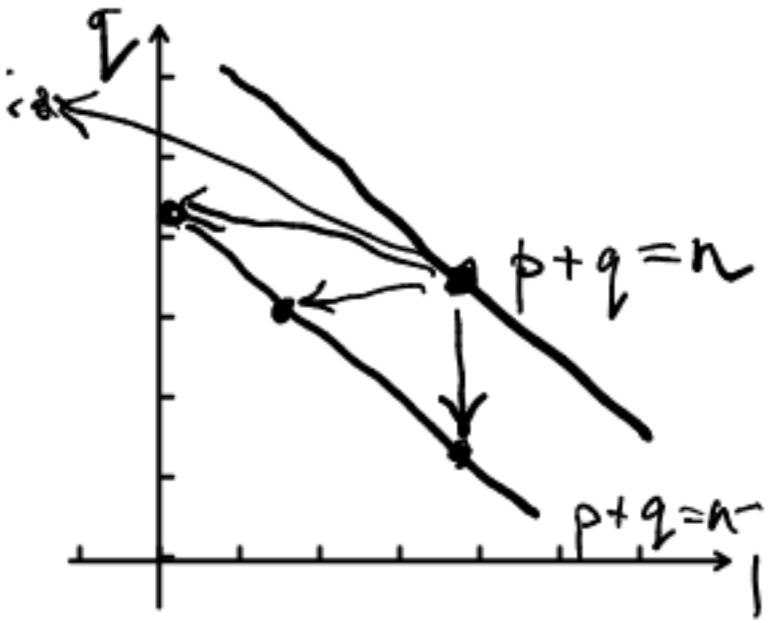


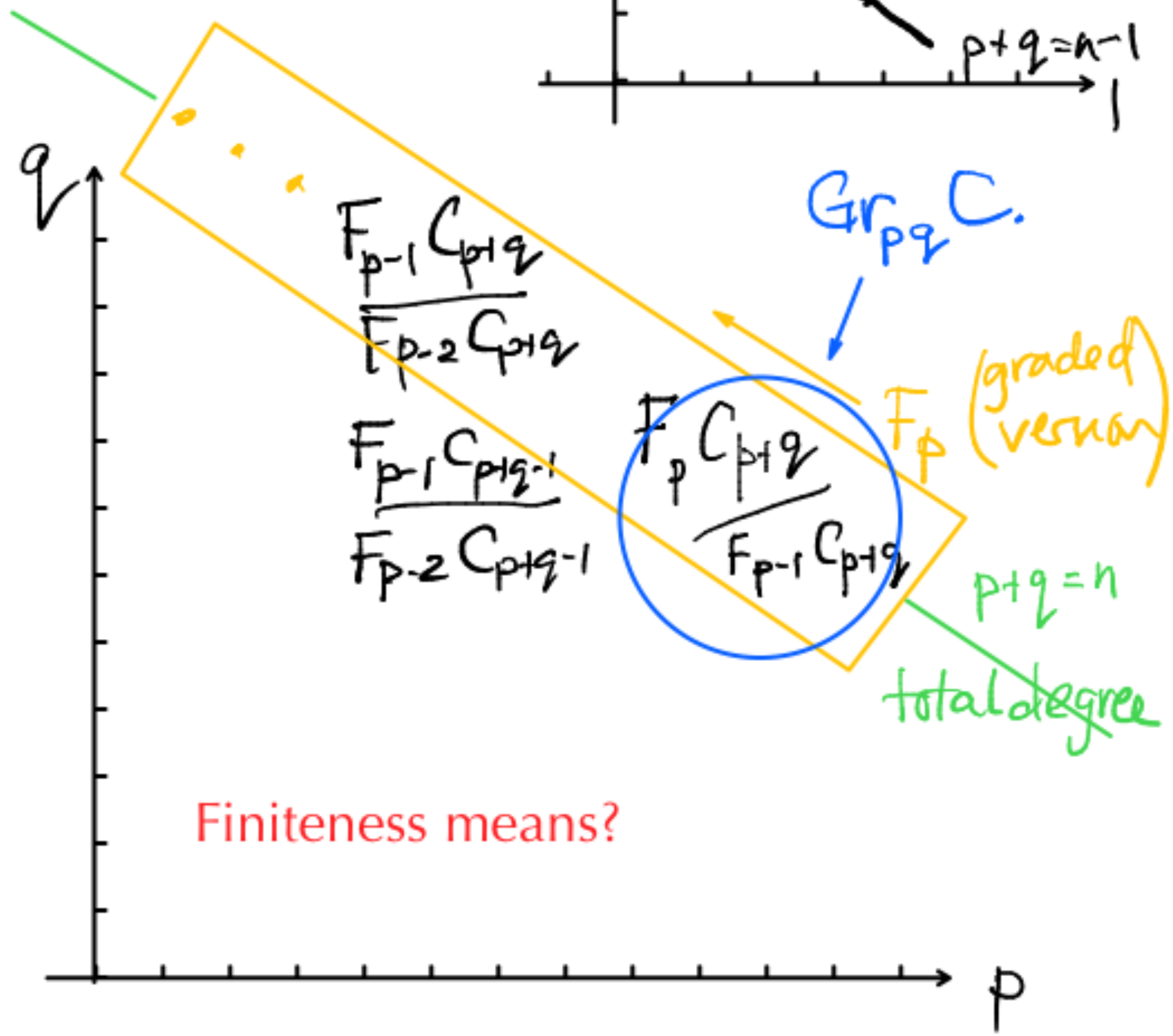
The associated graded object has $Gr_p C. = F_pC. / F_{p-1}C.$

which is also a list of modules indexed by homological degree n.

We consider filtrations finite in each homological degree.

The differential looks like





Finiteness means?

Pre-class Warm-up!!

Suppose we have a chain complex C. That is filtered

$$\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots$$

Writing the terms of the associated graded complex on a grid, as we did last time, where would we position the term of F_5 C / F_4 C that is in homological degree 7?

A At position (5,7)

B At position (4,7)

C At position (5,2)

D At position (7,5)

E None of the above.

homological degree of E_{pq} is $p+q$

p indexes the filtration.

How did that work for you?

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D Shaky

The filtration on the homology of a filtered complex.

Set up: we have a filtration $F_pC \subseteq F_{p+1}C \subseteq \cdots \subseteq C$. The inclusion $F_pC \xrightarrow{\iota} C$ gives a map in homology $H_*(F_p(C)) \xrightarrow{\iota_*} H_*(C)$.

Define $F_p(H_*(C)) = \text{Image of } \iota_*$.

This gives a filtration of $H_*(C)$ and an associated graded group.

Write $H_*(C) = Z/B$ suppressing homological degree

Proposition.

a. The image of H.(F_p C) in H.(C) is

$$(F_pC \cap Z)/(F_pC \cap B)$$

$$Gr_p H(C) = (F_pC \cap Z)/((F_pC \cap B) + (F_{p-1}C \cap Z))$$

Proof a. $H_*(F_pC) = \dfrac{\text{cycles of } F_pC}{\partial(F_pC)}$

$$= \frac{F_pC \cap Z}{\partial(F_pC)}$$

The map $H_*(F_pC) \to H_*C$ is induced by the $F_pC \cap Z \to Z/B$ and surjects to $F_p(H_*C)$.

The kernel is $(F_pC \cap Z) \cap B = F_pC \cap B$

b. $Gr_p(HC) = F_p(HC)/F_{p-1}(HC)$

$$= \frac{(F_pC \cap Z) + B}{B} \Big/ \frac{(F_{p-1}C \cap Z) + B}{B}$$

$$= \frac{(F_pC \cap Z) + B}{(F_{p-1}C \cap Z) + B} \cong \frac{F_pC \cap Z}{((F_{p-1}C \cap Z) + B) \cap (F_pC \cap Z)}$$

$$= \frac{F_pC \cap Z}{(F_{p-1}C \cap Z) + (B \cap F_pC)}$$ by the modular law

How did that work for you?

A I so totally got that

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D Shaky

The spectral sequence of a filtered complex

We define for each $r = 0, 1, 2, \ldots$

$$Z^r_{pq} = F_pC_{p+q} \cap \partial^{-1} F_{p-r} C_{p+q-1}$$

$$\partial : C_{p+q} \to C_{p+q-1}$$

$$Z^\infty_p = F_pC \cap Z$$

$$B^r_{pq} = F_pC_{p+q} \cap \partial F_{p+r-1} C_{p+q+1}$$

$$B^\infty_p = F_pC \cap B$$

Proposition.
Assume the filtration is finite in each homological degree. Then

$$B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^\infty_p \subseteq Z^\infty_p \subseteq \cdots \subseteq Z^1_p \subseteq Z^0_p = F_pC$$

In each degree the B and Z sequences stabilize.

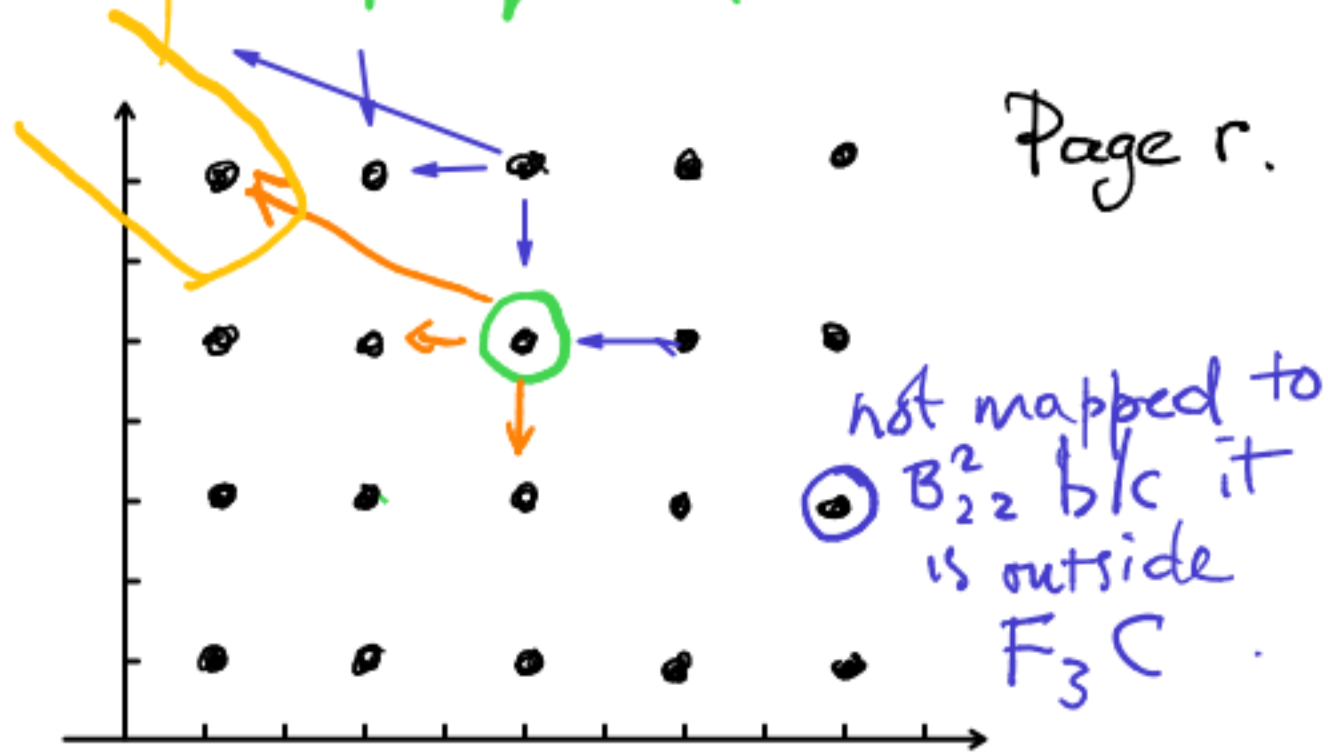
$Z^2_{2,2}$ = all elts of F_2C_4 that map into F_0C_3

$$B^2_{22} = \partial F_3 C_5$$

Proof B^r_p is the image of something bigger than B^{r-1}_p is. As r increases Z^r_p is the preimage of something smaller.

$$F_{p-r}C_{p+q-1} = F_0C_3$$

$r = 2 \quad p+q=4 \quad p=2 \quad \partial$



How did that work for you?

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Definition

$$E_p^r = Z_p^r / (B_p^r + Z_{p-1}^{r-1})$$

$$= Z_p^r / (B_p^r + (F_{p-1}C \cap Z_p^r))$$

$$E_p^\infty = Z_p^\infty / (B_p^\infty + Z_{p-1}^\infty) = Gr_p\, H(C)$$

Proposition.

a. $Z_{p-1}^{r-1} = F_{p-1}C \cap Z_p^r$

b. $Z_p^\infty / (B_p^\infty + Z_{p-1}^\infty) = Gr_p\, H(C)$

c. For fixed (p,q) we have

$$E_{pq}^r = E_{pq}^{r+1} = \cdots = E_{pq}^\infty$$

For r sufficiently large. The sequence 'converges' to Gr H(C) as r -> ∞.

Which seems hardest? A a. B b. C c.

Other terminology: H(C) is the abutment of the spectral sequence.

How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky

Question:

$E_p^r = Z_p^r / B_p^r$ and $E_p^\infty = Z_p^\infty / B_p^\infty$?

Proposition.
The E^0 and E^1 pages of the spectral sequence are as follows:

a. $E^0_p = F_pC / F_{p-1}C = Gr_p C$

b. $E^1_p = H(F_pC / F_{p-1}C)$

Thus E^1 is the homology of E^0, relative to the differential induced on E^0 by ∂

Where would you draw E_p^0 on the grid?

A the vertical line distance p from the origin.

B the horizontal line distance p from the origin

C the slope -1 line distance p from the origin.

D at coordinate (p,0)

Proposition. ∂ induces a differential on E^r of bidegree (-r, r-1) so that E^{r+1} = H(E^r).

Example. Consider a short exact sequence of chain complexes $0 \to A \to C \to B \to 0$.

Proposition Let $\mu : C \to C'$ be a filtration-preserving chain map, where C and C' have degree-wise finite filtrations. If the induced map $E^r(\mu) : E^r(C) \to E^r(C')$ of spectral sequences is an isomorphism for some r, then $H(\mu) : H(C) \to H(C')$ is an isomorphism.

Spectral sequences can be used to compute Euler characteristics using any of their pages.