Spectral sequences

Source: I prefer the treatment in
K.S. Brown, Cohomology of groups,
chapter VII

Topics:
- the spectral sequence of a filtered complex
- how these arise from double complexes
- application to the homology of a union of spaces.

Motivation

We know that a short exact sequence of chain complexes 0 -> A. -> B. -> C. -> 0 gives rise to a long exact sequence in homology, perhaps giving information about H_*(B.)

Examples

1. Ext groups

Given a s.e.s. of R-modules $0 \to L \to M \to N \to 0$ we get a s.e.s of chain complexes

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0$$

where $P \to A \to 0$ is a proj. resol'n of A, hence a long e.s.

$$0 \to \mathrm{Ext}^0_R(A, L) \to \qquad \to$$

$\to$ . . . etc.

2. We may have a simplicial complex $X \cup Y$ where $X \cap Y$ is a subsimplicial complex

We have a s.e.s. of chain cxes

$$0 \to C_\bullet(X \cap Y) \to C_\bullet(X) \oplus C_\bullet(Y) \to C_\bullet(X \cup Y) \to 0$$

Get long e.s. in homology.

What if the simplicial complex Δ has several subcomplexes

$X_1, \dots, X_n$.

$\Delta = \bigcup X_i$.

$C_\cdot(X_i) \subseteq C_\cdot(\Delta)$.

Let $F_p(\Delta)$ = span of the simplices in Δ that lie in at least p of the $X_1, \dots, X_n$.

We get subcomplexes

$$\cdots F_3(\Delta) \subseteq F_2(\Delta) \subseteq F_1(\Delta) \subseteq F_0(\Delta) = C_\cdot(\Delta)$$

Can we get info about

$H_*(C_\cdot(\Delta))$

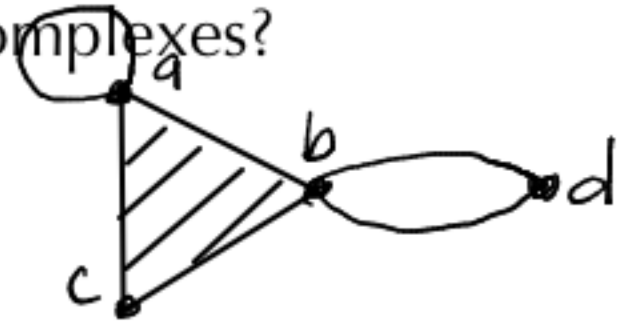
from the $H_*\left(F_p(\Delta)/F_{p+1}(\Delta)\right)$?

Yes!

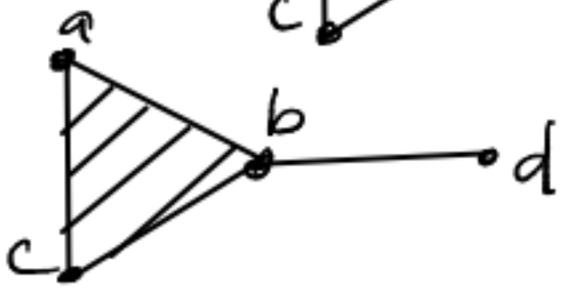
There is a spectral sequence generalizing the Mayer–Vietoris long e. s.

Pre-class Warm-up!!

Which of the following define the same simplicial complexes?

1. 

Not a simplicial complex

2. 

3. $\{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}\}$

A 1 and 2 describe the same simplicial complex.

B 1 and 3 describe the same simplicial complex.

C 2 and 3 describe the same simplicial complex.

D They all describe the same simplicial complex.

An (abstract) simplicial complex is a set Δ of subsets of a set S so that $T \in \Delta$, $U \subseteq T$ $\Rightarrow$ $U \in \Delta$.

Filtrations of modules and associated graded modules

An ascending filtration of a module M is a chain of submodules

$$\cdots \subseteq F_p(M) \subseteq F_{p+1} \subseteq \cdots \subseteq M.$$

A $\mathbb{Z}$-graded module is a list of modules M_p, $p \in \mathbb{Z}$.
We may want to think of it as

$$\bigoplus_{p \in \mathbb{Z}} M_p.$$

Given a filtration the associated graded module $\mathrm{Gr}\, M$ has

$$\mathrm{Gr}_p M = F_p M / F_{p-1} M.$$

e.g. $k[x] = \bigoplus_{p \geq 0} k x^p$

We assume that filtrations are finite.

This means $F_p = F_{p+1} = \cdots$ if p is large enough. and

$F_p = F_{p-1} = \cdots$ if p is small enough.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

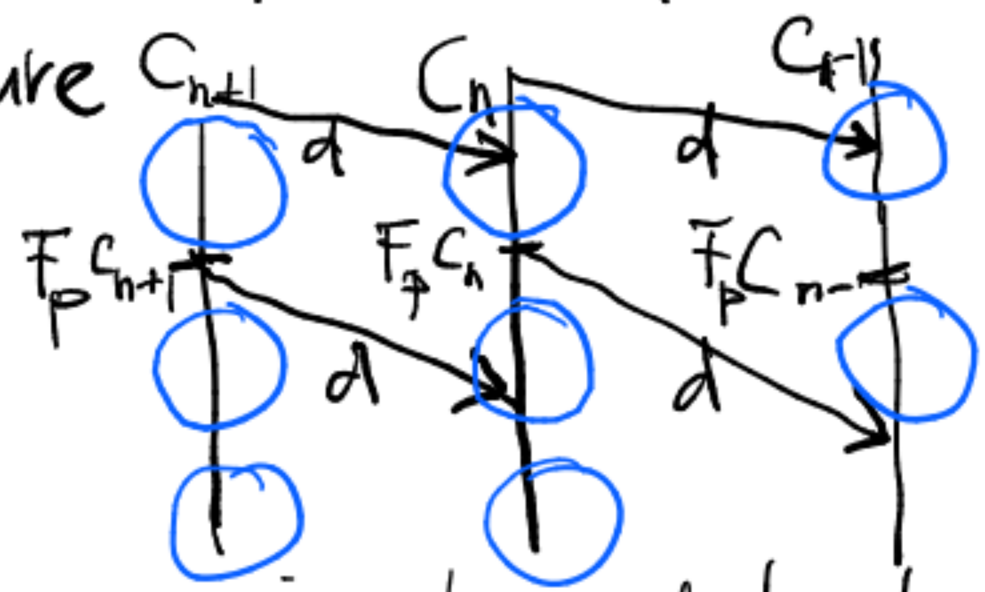
D Shaky

Definition.

A filtration of a chain complex $C_\cdot$ is a ~~chain~~ list of subcomplexes

$$\cdots \subseteq F_p(C_\cdot) \subseteq F_{p+1}(C_\cdot) \subseteq \cdots$$

Picture



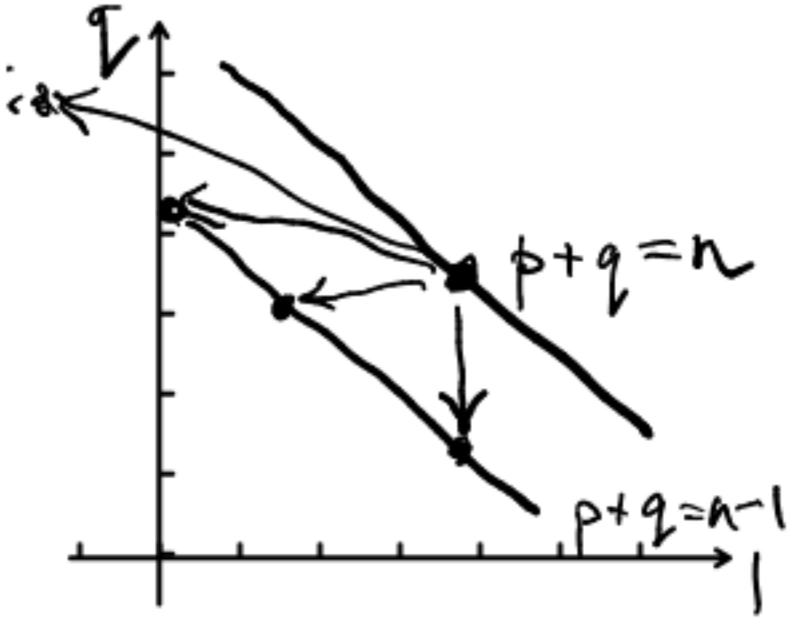
The associated graded object has

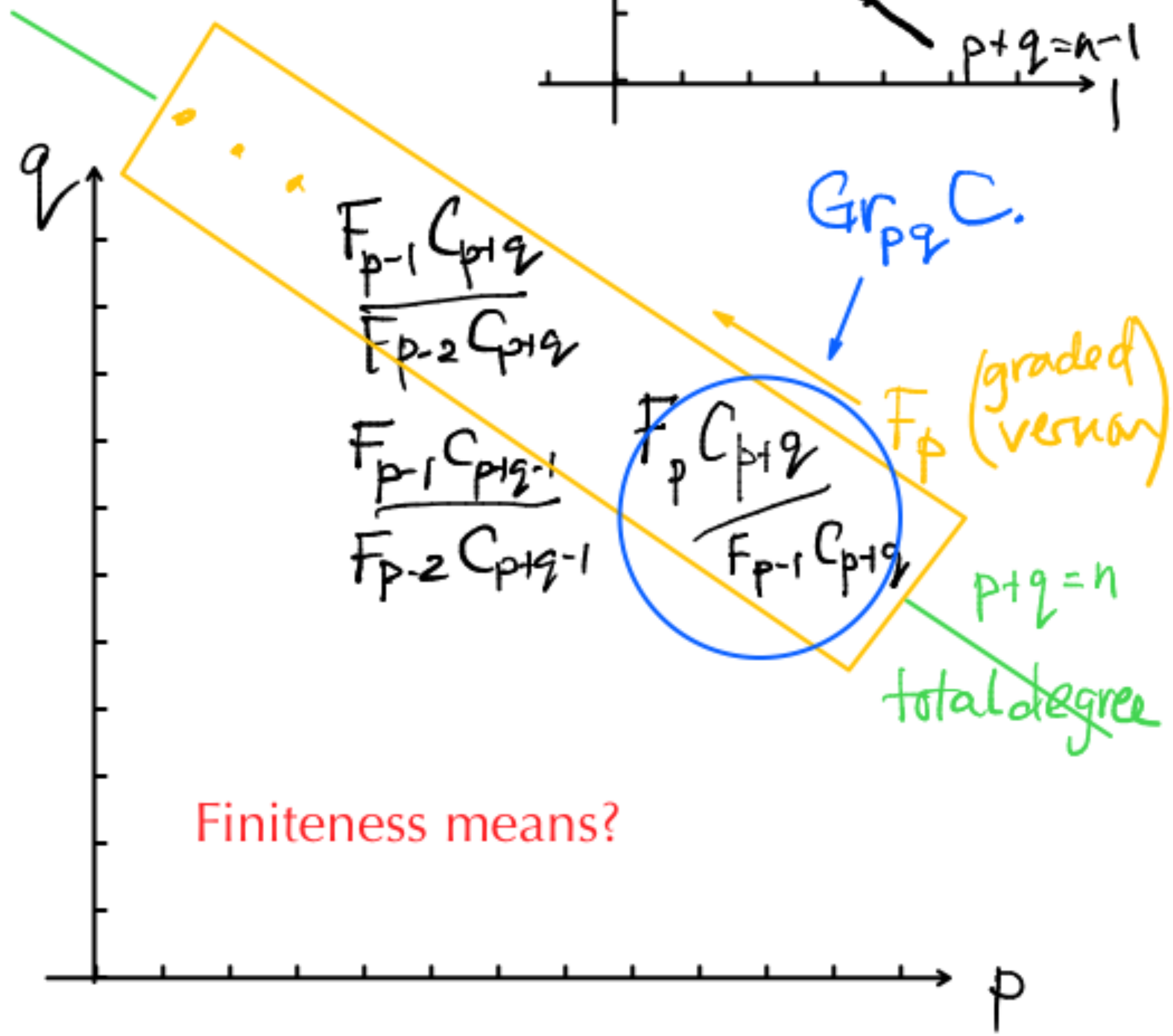
$$Gr_p C_\cdot = F_pC_\cdot / F_{p-1}C_\cdot$$

which is also a list of modules indexed by homological degree n.

We consider filtrations finite in each homological degree.

The differential looks like





Pre-class Warm-up!!

Suppose we have a chain complex C. That is filtered

$$\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots$$

Writing the terms of the associated graded complex on a grid, as we did last time, where would we position the term of F_5 C / F_4 C that is in homological degree 7?

A At position (5,7)

B At position (4,7)

C At position (5,2)

D At position (7,5)

E None of the above.

homological degree of E_{pq} is $p+q$

p indexes the filtration.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

D Shaky

The filtration on the homology of a filtered complex.

Set up: we have a filtration $F_pC \subseteq F_{p+1}C \subseteq \cdots \subseteq C$. The inclusion $F_pC \xrightarrow{\iota} C$ gives a map in homology $H_*(F_p(C)) \xrightarrow{\iota_*} H_*(C)$.

Define $F_p(H_*(C)) = $ Image of ι_*.

This gives a filtration of $H_*(C)$ and an associated graded group.

Write $H_*(C) = Z/B$ suppressing homological degree

Proposition.

a. The image of H.(F_p C) in H.(C) is $(F_pC \cap Z)/(F_pC \cap B)$

$$Gr_p\, H(C) = (F_p C \cap Z)/((F_pC \cap B) + (F_{p-1} C \cap Z))$$

Proof a.

$$H_*(F_pC) = \frac{\text{cycles of } F_pC}{\partial(F_pC)} = \frac{F_pC \cap Z}{\partial(F_pC)}$$

The map $H_*(F_pC) \to H_*C$ is induced by the $F_pC \cap Z \to Z/B$ and surjects to $F_p(H_*C)$.

The kernel is $(F_pC \cap Z) \cap B = F_pC \cap B$

b. $Gr_p(HC) = F_p(HC)/F_{p-1}(HC)$

$$= \frac{(F_pC \cap Z)+B}{B} \Big/ \frac{(F_{p-1}C \cap Z)+B}{B}$$

$$= \frac{(F_pC \cap Z)+B}{(F_{p-1}C \cap Z)+B} \cong \frac{F_pC \cap Z}{((F_{p-1}C \cap Z)+B) \cap (F_pC \cap Z)}$$

$$= \frac{F_pC \cap Z}{(F_{p-1}C \cap Z) + (B \cap F_pC)}$$ by the modular law

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

D Shaky

The spectral sequence of a filtered complex

We define for each $r = 0, 1, 2, \ldots$

$$Z^r_{pq} = F_p C_{p+q} \cap \partial^{-1} F_{p-r} C_{p+q-1}$$

$\partial : C_{p+q} \to C_{p+q-1}$

$$Z^\infty_p = F_p C \cap Z$$

$$B^r_{pq} = F_p C_{p+q} \cap \partial F_{p+r-1} C_{p+q+1}$$

$$B^\infty_p = F_p C \cap B$$

Pre-class Warm-up!
Study this setup and get familiar with it.

Proposition.
Assume the filtration is finite in each homological degree. Then

$$B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^\infty_p \subseteq Z^\infty_p \subseteq \cdots$$
$$\subseteq Z^1_p \subseteq Z^0_p = F_p C$$

In each degree the B and Z sequences stabilize.

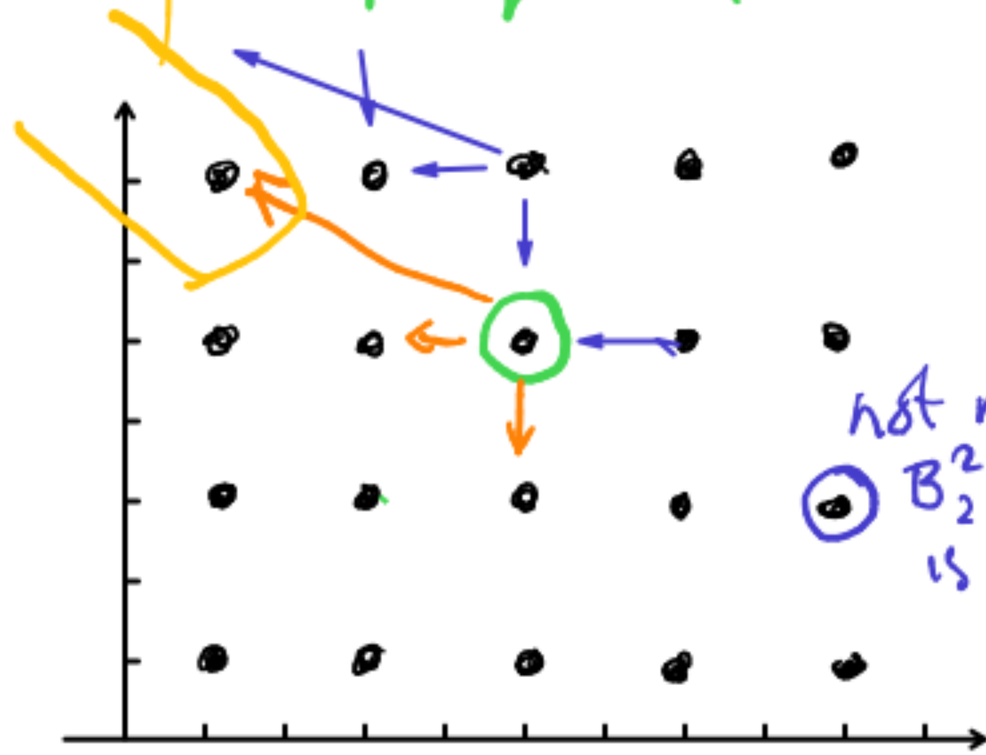
$Z^2_{2,2}$ = all elts of $F_2 C_4$ that map into $F_0 C_3$

$$B^2_{22} = \partial F_3 C_5$$

Proof B^r_p is the image of something bigger than B^{r-1}_p is. As r increases Z^r_p is the preimage of something smaller.

$F_{p-r} C_{p+q-1} = F_0 C_3$

$r = 2 \quad p+q = 4 \quad p = 2 \quad \partial$



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not mapped to B^2_{22} b/c it is outside $F_3 C$.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

D Shaky

Definition

$$E^r_{pq} = Z^r_{pq}/(B^r_{pq} + Z^{r-1}_{p-1q})$$

$$= Z^r_p/(B^r_p + (F_{p-1}C \cap Z^r_p))$$

$$E^\infty_{pq} = Z^\infty_{pq}/(B^\infty_{pq} + Z^\infty_{p-1q}) = Gr_p\, H(C)_{p+q}$$

Proposition.

a. $Z^{r-1}_{p-1} = F_{p-1}C \cap Z^r_p$

b. $Z^\infty_p/(B^\infty_p + Z^\infty_{p-1}) = Gr_p\, H(C)$

c. For fixed (p,q) we have

$$E^r_{pq} = E^{r+1}_{pq} = \cdots = E^\infty_{pq}$$

For r sufficiently large. The sequence 'converges' to $Gr\, H(C)$ as $r \to \infty$.

Which seems hardest? A a. B b. C c.

Proof a.

$Z^{r-1}_{p-1} = F_{p-1}C \cap Z^r_p$ b/c

Z^r_p = those x in F_pC, $\partial x \in F_{p-r}C$

Z^{r-1}_{p-1} = those x in $F_{p-1}C$, $\partial x \in F_{p-r}C$

b. $Gr_p\, H(C) = (F_pC \cap Z)/[(F_pC \cap B) + (F_{p-1}C \cap Z)]$

$$\underbrace{(F_pC \cap Z)}_{Z^\infty_p} \quad \underbrace{(F_pC \cap B)}_{B^\infty_p} \quad \underbrace{(F_{p-1}C \cap Z)}_{Z^\infty_{p-1}}$$

c. The terms $B^r_p \subseteq B^{r+1}_p \subseteq B^\infty_p$ stabilize with r. So do the Z^r_p.

Assume F_pC stabilizes in each homological degree at C. Then we get $= E^\infty_{pq}$

Other terminology: $H(C)$ is the abutment of the spectral sequence.

How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky

Question:

$E^r_p = Z^r_p / B^r_p$ and $E^\infty_p = Z^\infty_p / B^\infty_p$?

Proposition.
The E^0 and E^1 pages of the spectral sequence are as follows:

a. $E^0_p = F_pC / F_{p-1}C = Gr_p C$

b. $E^1_p = H_\cdot(F_pC/F_{p-1}C)$

Thus E^1 is the homology of E^0, relative to the differential induced on E^0 by ∂

Proof. a. $E^0_p = Z^0_p / (B^0_p + Z^{-1}_{p-1})$

We take $Z^{-1}_{p-1} = Z^0_{p-1} \doteq F_{p-1}C$

$B^0_p = F_pC \cap \partial F_{p-1}C \subseteq F_{p-1}C$.

$E^0_p = F_pC / F_{p-1}C$

b. $E^1_p = Z^1_p / (B^1_p + Z^0_{p-1})$

$= (F_pC \cap \partial^{-1}F_{p-1}C) / ((\partial F_pC \cap F_pC) + F_{p-1}C)$

Where would you draw E_p^0 on the grid?

A the vertical line distance p from the origin.

B the horizontal line distance p from the origin

C the slope -1 line distance p from the origin.

D at coordinate (p,0)

= things sent to 0 in $F_pC/F_{p-1}C$ by ∂ / Image of $\partial : F_pC/F_{p-1}C \to F_pC/F_{p-1}C$

$= H_\cdot(F_pC/F_{p-1}C)$

Proposition. ∂ induces a differential on E^r of bidegree (-r, r-1) so that E^{r+1} = H(E^r).

Example. Consider a short exact sequence of chain complexes 0 -> A -> C -> D -> 0.

$A_0 \xleftarrow{\partial} A_1 \leftarrow$

This means we have a filtration of C.

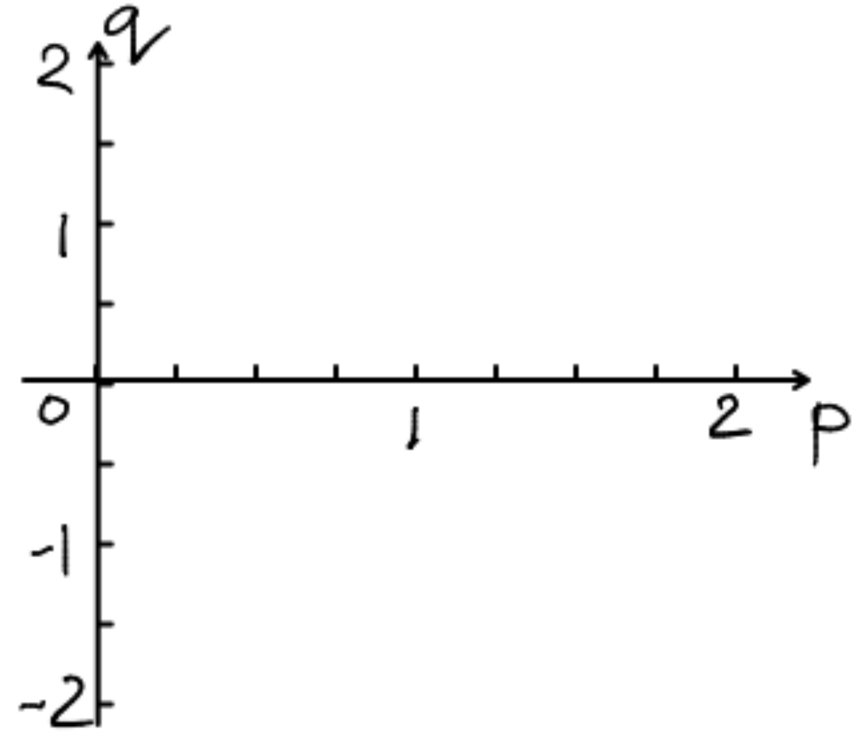
$F_0 C = 0 \quad F_1 C = A. \quad F_2 C = C.$

so $F_2 C / F_1 C = D.$

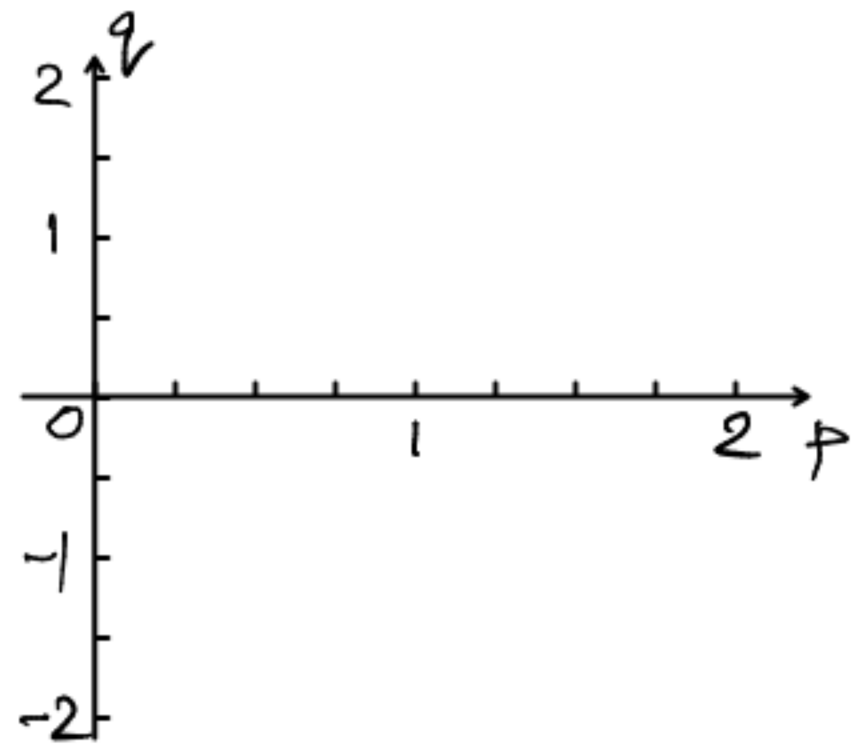
We draw the E^0 page of the spectral sequence

$q \backslash p$	0	1	2
2	0	A_3	D_4
1	0	A_2	D_3
0	0	A_1	D_2
-1	0	A_0	D_1
-2			D_0

The E^1 page.



The E^∞ page.



Proposition Let $\mu : C \to C'$ be a filtration-preserving chain map, where C and C' have degree-wise finite filtrations. If the induced map $E^r(\mu) : E^r(C) \to E^r(C')$ of spectral sequences is an isomorphism for some r, then $H(\mu) : H(C) \to H(C')$ is an isomorphism.

Spectral sequences can be used to compute Euler characteristics using any of their pages.

Double complexes

Definition
A double complex is a bigraded module $C =$

With a `horizontal′ differential $\partial'$ of bidegree (-1,0) and a `vertical′ differential ∂'' of bidegree (0,-1) so that

The total complex TC has

Example. The tensor product of two chain complexes C' and C''