

## Spectral sequences

Source: I prefer the treatment in  
K.S. Brown, Cohomology of groups,  
chapter VII

Topics:

- the spectral sequence of a filtered complex
- how these arise from double complexes
- application to the homology of a union of spaces.

## Motivation

We know that a short exact sequence of chain complexes  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  gives rise to a long exact sequence in homology, perhaps giving information about  $H_*(B)$

## Examples 1. Ext groups

Given a s.e.s. of  $R$ -modules

$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  we get a s.e.s. of chain complexes

$$0 \rightarrow \text{Hom}(P, L) \rightarrow \text{Hom}(P, M) \rightarrow \text{Hom}(P, N) \rightarrow 0$$

where  $P \rightarrow A \rightarrow 0$  is a proj. res. of  $A$ , hence a long e.s.

$$0 \rightarrow \text{Ext}_R^0(A, L) \rightarrow \dots \rightarrow \dots \text{etc.}$$

2. We may have a simplicial complex  $X \cup Y$  where  $X \cap Y$  is a subsimplicial complex



We have a s.e.s. of chain cxs

$$0 \rightarrow C_*(X \cap Y) \rightarrow C_*(X) \oplus C_*(Y) \rightarrow C_*(X \cup Y) \rightarrow 0$$

Get long e.s. in homology.

What if the simplicial complex  $\Delta$  has several subcomplexes  $X_1, \dots, X_n$ .

$$\Delta = \cup X_i$$

$$C.(X_i) \subseteq C.(\Delta)$$

Let  $F_p(\Delta) =$  span of the simplices in  $\Delta$  that lie in at least  $p$  of the  $X_1, \dots, X_n$ .

We get subcomplexes

$$\dots F_3(\Delta) \subseteq F_2(\Delta) \subseteq F_1(\Delta) \subseteq F_0(\Delta) = C.(\Delta)$$

Can we get info about

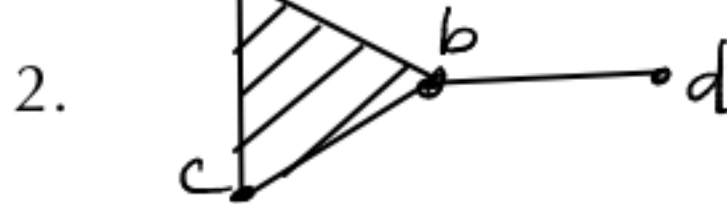
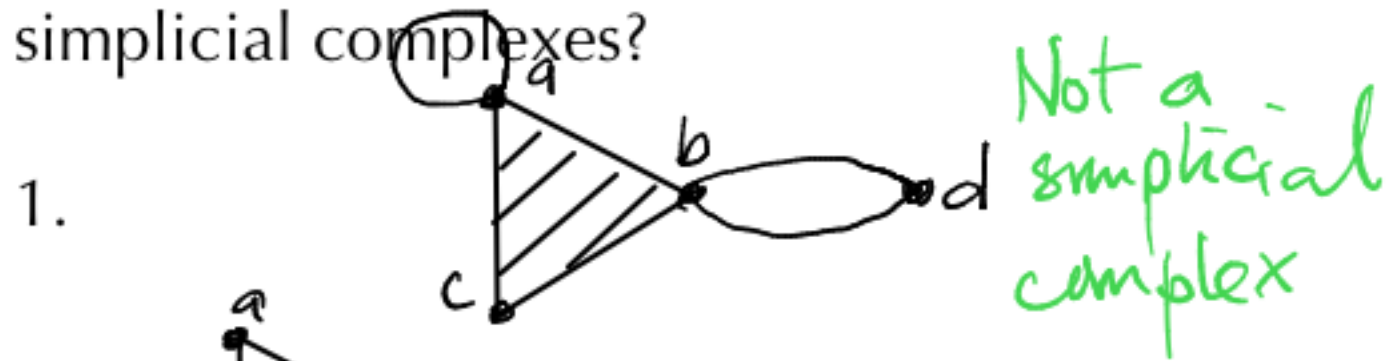
$H_*(C.(\Delta))$   
from the  $H_*(F_p(\Delta)/F_{p+1}(\Delta))$ ?

Yes!

There is a spectral sequence generalizing the Mayer-Vietoris long e. s.

# Pre-class Warm-up!!

Which of the following define the same simplicial complexes?



3.  $\left\{ \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\} \right\}$

A 1 and 2 describe the same simplicial complex.

B 1 and 3 describe the same simplicial complex.

C 2 and 3 describe the same simplicial complex.

D They all describe the same simplicial complex.

An (abstract) simplicial complex is a set  $\Delta$  of subsets of a set  $S$  so that  $T \in \Delta, U \subseteq T \Rightarrow U \in \Delta$ .

## Filtrations of modules and associated graded modules

An ascending filtration of a module  $M$  is a chain of submodules

$$\dots \subseteq F_p(M) \subseteq F_{p+1} \subseteq \dots \subseteq M.$$

A  $\mathbb{Z}$ -graded module is a list of modules  $M_p$ ,  $p \in \mathbb{Z}$ . We may want to think of it

$$\text{as } \bigoplus_{p \in \mathbb{Z}} M_p.$$

Given a filtration the associated graded module  $\text{Gr } M$  has

$$\text{Gr}_p M = F_p M / F_{p-1} M.$$

$$\text{e.g. } k[x] = \bigoplus_{p \geq 0} kx^p$$

We assume that filtrations are finite.

This means  $F_p = F_{p+1} = \dots$   
if  $p$  is large enough, and

$F_p = F_{p-1} = \dots$  if  $p$  is small enough.

How did that work for you?

A I so totally got that

B OK

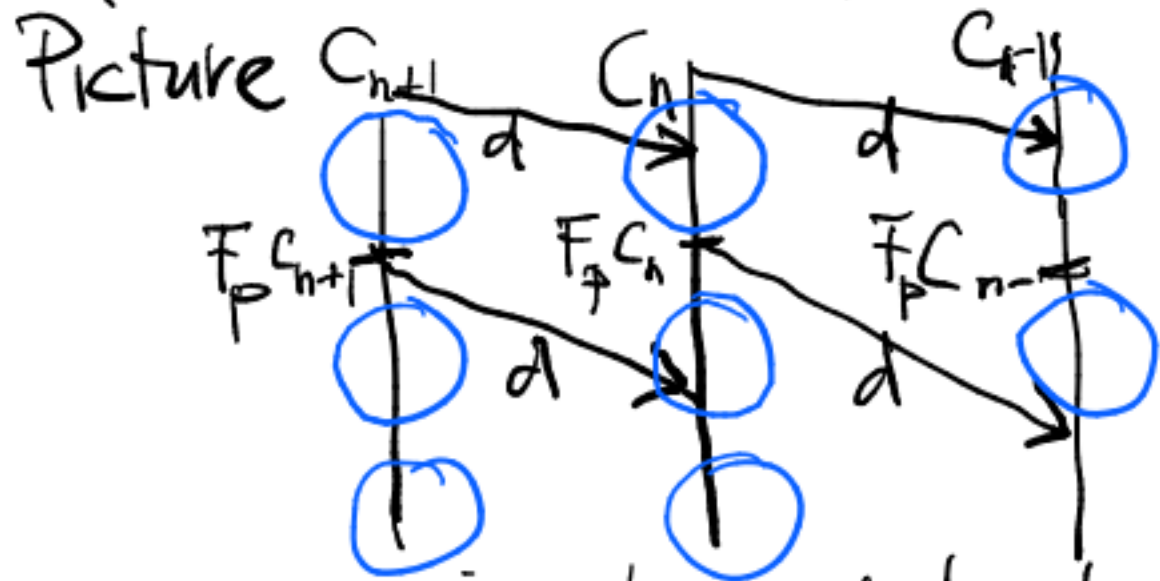
C I'm not sure about what we just did.

D Shaky

Definition.

A filtration of a chain complex  $C$  is a chain of subcomplexes

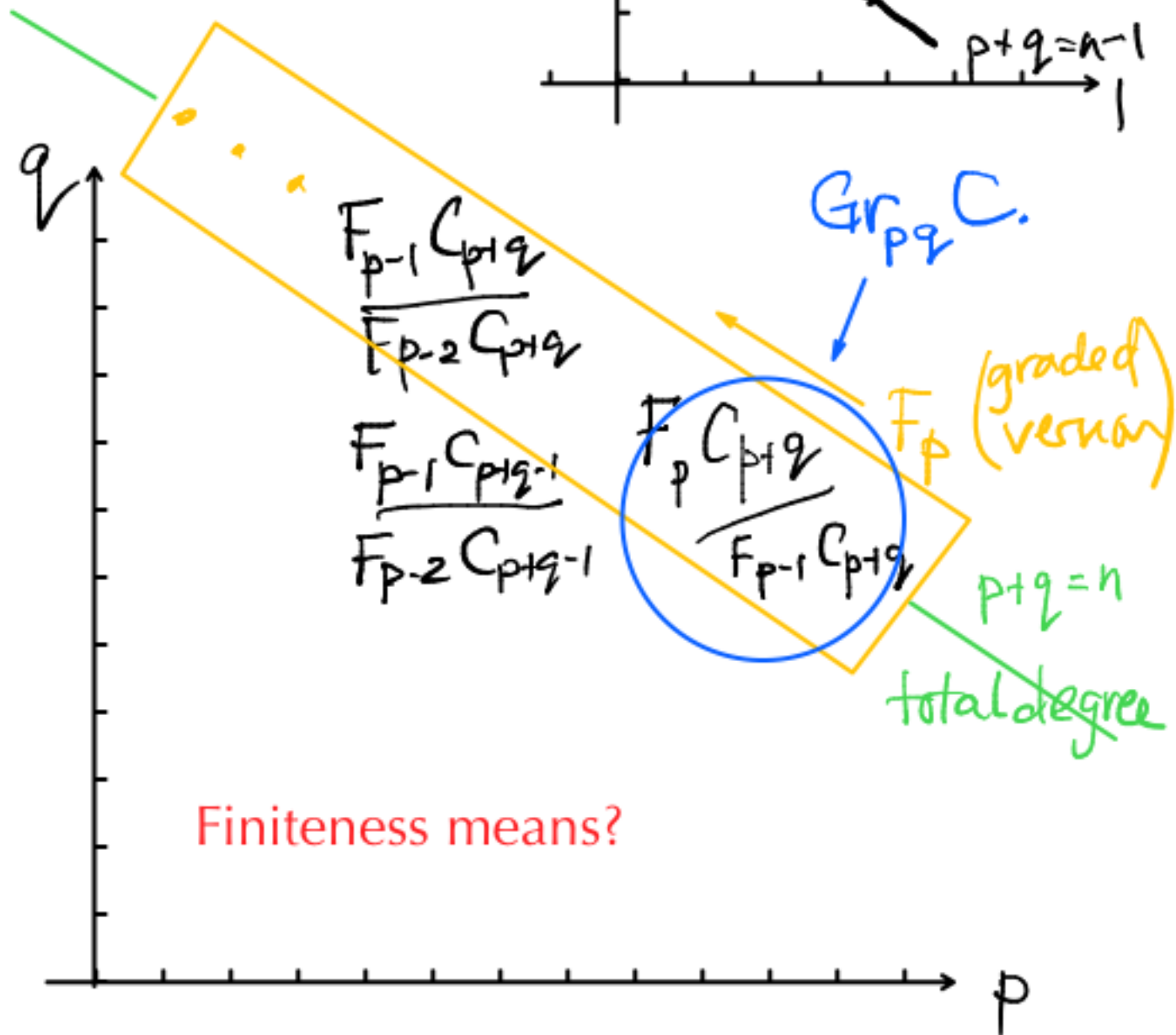
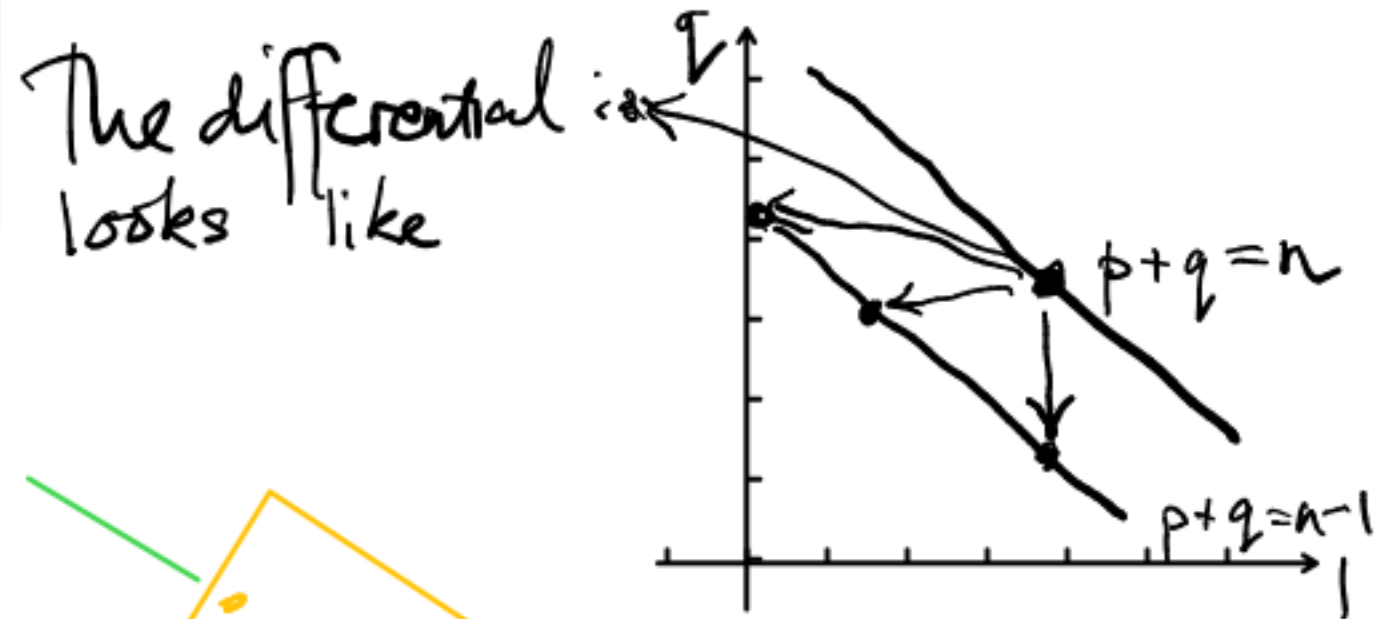
$$\dots \subseteq F_p(C) \subseteq F_{p+1}(C) \subseteq \dots$$



The associated graded object has  $Gr_p C = F_p C / F_{p-1} C$ .

which is also a list of modules indexed by homological degree  $n$ .

We consider filtrations finite in each homological degree.



# Pre-class Warm-up!!

Suppose we have a chain complex  $C$ . That is filtered

$$\cdots \subseteq F_{p-1}C \subseteq F_p C \subseteq F_{p+1}C \subseteq \cdots$$

Writing the terms of the associated graded complex on a grid, as we did last time, where would we position the term of  $F_5 C / F_4 C$  that is in homological degree 7?

A At position (5,7)

B At position (4,7)

C At position (5,2)

D At position (7,5)

E None of the above.

homological degree  
of  $E_{pq}$  is  $p+q$   
 $p$  indexes the  
filtration.

How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky



The filtration on the homology of a

filtered complex. Set up: we have a filtration

$$F_p C \subseteq F_{p+1} C \subseteq \dots \subseteq C. \text{ The inclusion}$$

$F_p C \xrightarrow{L} C$  gives a map in homology

$$H_*(F_p C) \xrightarrow{L_*} H_*(C).$$

Define  $F_p(H_*(C)) = \text{Image of } L_*$ .

This gives a filtration of  $H_*(C)$  and an associated graded group.

Write  $H_*(C) = Z/B$  suppressing homological degree

Proposition.

a. The image of  $H_*(F_p C)$  in  $H_*(C)$  is

$$(F_p C \cap Z) / (F_p C \cap B)$$

$$\text{Gr}_p H(C) = (F_p C \cap Z) / ((F_p C \cap B) + (F_{p-1} C \cap Z))$$

Proof a.

$$H_*(F_p C) = \frac{\text{cycles of } F_p C}{\partial(F_p C)}$$

$$= \frac{F_p C \cap Z}{\partial(F_p C)}$$

The map  $H_*(F_p C) \rightarrow H_*(C)$  is induced by the  $F_p C \cap Z \rightarrow Z/B$  and surjects to  $F_p(H_*(C))$ .

The kernel is  $(F_p C \cap Z) \cap B = F_p C \cap B$

b.  $\text{Gr}_p H(C) = F_p(H(C)) / F_{p-1}(H(C))$

$$= \frac{(F_p C \cap Z) + B}{B} / \frac{(F_{p-1} C \cap Z) + B}{B}$$

$$= \frac{(F_p C \cap Z) + B}{(F_{p-1} C \cap Z) + B} \stackrel{\cong}{\sim} \frac{F_p C \cap Z}{(F_{p-1} C \cap Z) + B / (F_p C \cap Z)}$$

$$= \frac{F_p C \cap Z}{(F_{p-1} C \cap Z) + (B \cap F_p C)} \text{ by the modular law}$$

How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky

The spectral sequence of a filtered complex

We define for each  $r = 0, 1, 2, \dots$

$$Z_{pq}^r = F_p C_{p+q} \cap \partial^{-1} F_{p-r} C_{p+q-1}$$

$$Z_p^\infty = F_p C \cap Z$$

$\partial: C_{p+q} \rightarrow C_{p+q-1}$

$$B_{pq}^r = F_p C_{p+q} \cap \partial F_{p+r-1} C_{p+q+1}$$

$$B_p^\infty = F_p C \cap B$$

Pre-class Warm-up!  
Study this setup and  
get familiar with it.

Proposition.

Assume the filtration is finite in each homological degree. Then

$$B_p^0 \subseteq B_p^1 \subseteq \dots \subseteq B_p^\infty \subseteq Z_p^\infty \subseteq \dots$$

$$\subseteq Z_p^1 \subseteq Z_p^0 = F_p C$$

In each degree the  $B$  and  $Z$  sequences stabilize.

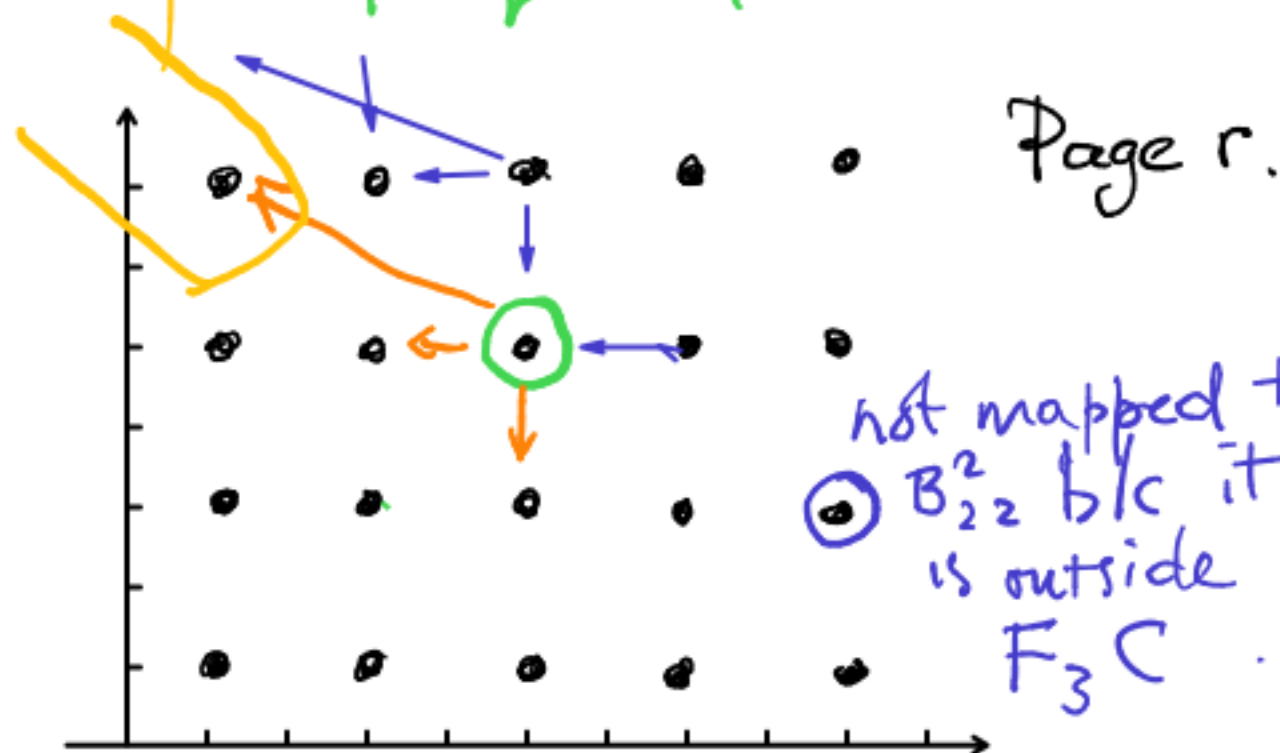
$Z_{2,2}^2 =$  all elts of  $F_2 C_4$  that map into  $F_0 C_3$

$$B_{2,2}^2 = \partial F_3 C_5$$

Proof  $B_p^r$  is the image of something bigger than  $B_p^{r-1}$  is. As  $r$  increases  $Z_p^r$  is the preimage of something smaller.

$$F_{p-r} C_{p+q-1} = F_0 C_3$$

$r=2 \quad p+q=4 \quad p=2 \quad \partial$



How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky

# Definition

$$E_{pq}^r = Z_{pq}^r / (B_{pq}^r + Z_{p-1,q}^{r-1})$$

$$= Z_p^r / (B_p^r + (F_{p-1}C \cap Z_p^r))$$

$$E_{pq}^\infty = Z_{pq}^\infty / (B_{pq}^\infty + Z_{p-1,q}^\infty) = \text{Gr}_p H(C)_{p+q}$$

# Proposition.

a.  $Z_{p-1}^{r-1} = F_{p-1}C \cap Z_p^r$

b.  $Z_p^\infty / (B_p^\infty + Z_{p-1}^\infty) = \text{Gr}_p H(C)$

c. For fixed  $(p,q)$  we have

$$E_{pq}^r = E_{pq}^{r+1} = \dots = E_{pq}^\infty$$

For  $r$  sufficiently large. The sequence 'converges' to  $\text{Gr} H(C)$  as  $r \rightarrow \infty$ .

Which seems hardest? A a. B b. C c.

# Proof a.

$$Z_{p-1}^{r-1} = F_{p-1}C \cap Z_p^r \quad \text{b/c}$$

$Z_p^r =$  those  $x$  in  $F_p C$ ,  $\partial x \in F_{p-r} C$

$Z_{p-1}^{r-1} =$  those  $x$  in  $F_{p-1} C$ ,  $\partial x \in F_{p-r} C$

$$\text{b. } \text{Gr}_p H(C) = (F_p C \cap Z) / (F_p C \cap B + (F_{p-1} C \cap Z))$$

$$= Z_p^\infty / (B_p^\infty + Z_{p-1}^\infty)$$

c. The terms  $B_p^r \subseteq B_p^{r+1} \subseteq B_p^\infty$  stabilize with  $r$ . So do the  $Z_p^r$ .

## Assume

$F_p C$  stabilizes in each homological degree at  $C$ . Then we get  $= E_{pq}^\infty$

Other terminology:  $H(C)$  is the abutment of the spectral sequence.

How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky

Question:

$$E_p^r = Z_p^r / B_p^r \text{ and } E_p^\infty = Z_p^\infty / B_p^\infty ?$$

Proposition.

The  $E^0$  and  $E^1$  pages of the spectral sequence are as follows:

$$a. E_p^0 = F_p C / F_{p-1} C = Gr_p C$$

$$b. E_p^1 = H_*(F_p C / F_{p-1} C)$$

Thus  $E^1$  is the homology of  $E^0$ , relative to the differential induced on  $E^0$  by  $\partial$

Proof. a.  $E_p^0 = Z_p^0 / (B_p^0 + Z_{p-1}^{-1})$

We take  $Z_{p-1}^{-1} = Z_{p-1}^0 \cong F_{p-1} C$

$$B_p^0 = F_p C \cap \partial F_{p-1} C \subseteq F_{p-1} C$$

$$E_p^0 = F_p C / F_{p-1} C$$

$$b. E_p^1 = Z_p^1 / (B_p^1 + Z_{p+1}^0)$$

$$= (F_p C \cap \partial^{-1} F_{p-1} C) / (\partial F_p C \cap F_p C) + F_{p-1} C$$

Where would you draw  $E_{-p}^0$  on the grid?

A the vertical line distance  $p$  from the origin.

B the horizontal line distance  $p$  from the origin

C the slope -1 line distance  $p$  from the origin.

D at coordinate  $(p,0)$

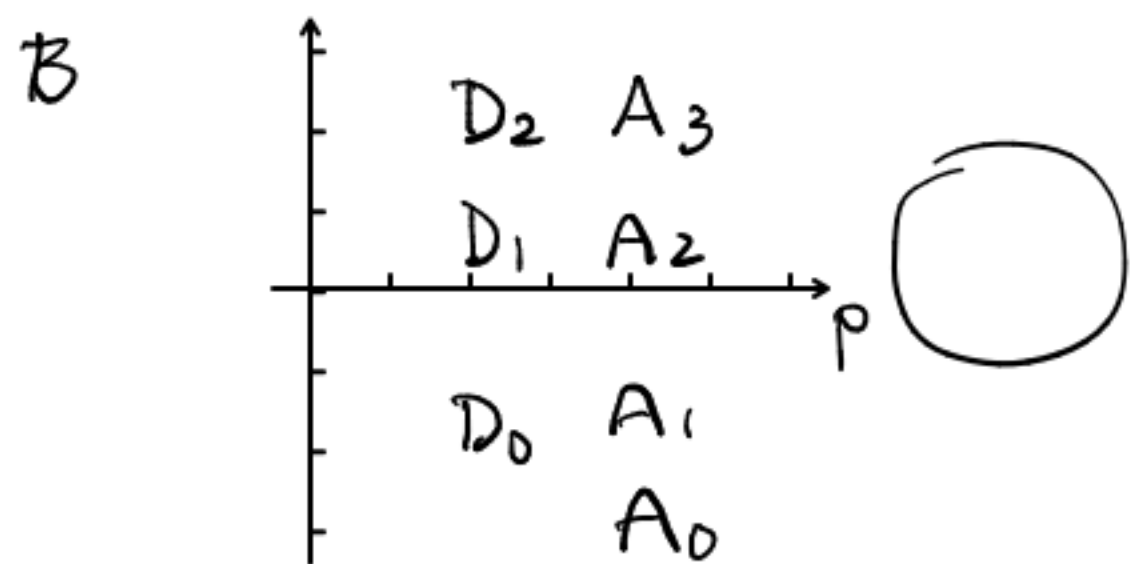
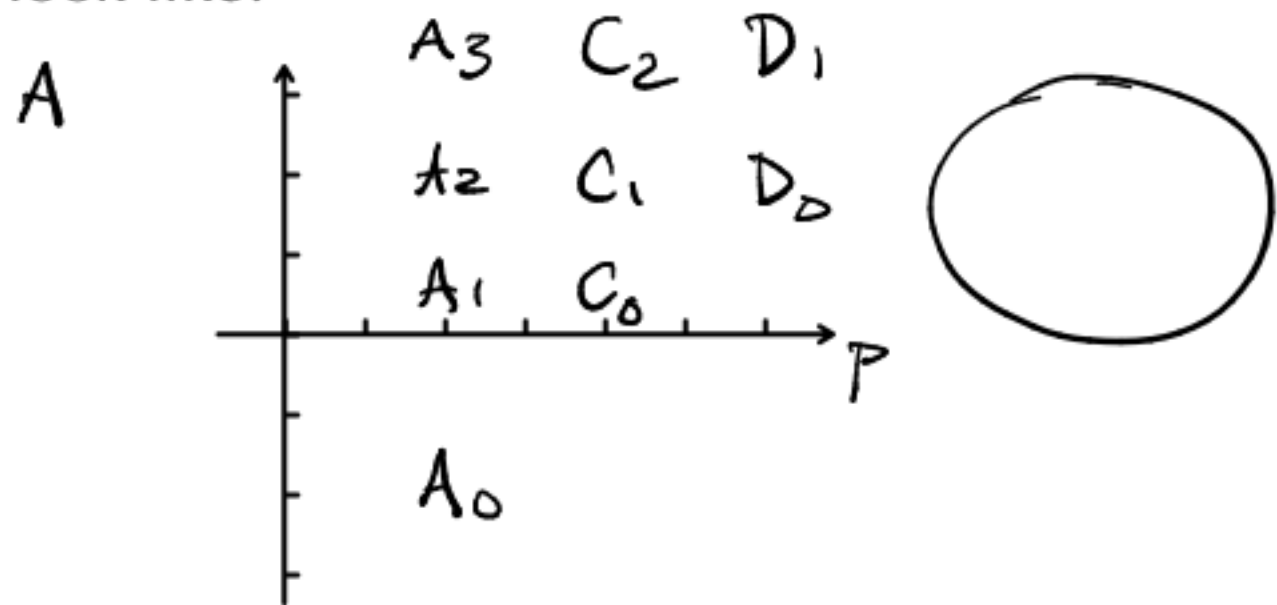
$$= \text{things sent to } 0 \text{ in } F_p C / F_{p-1} C / \text{Image of } \partial: F_p C / F_{p-1} C \rightarrow F_p C / F_{p-1} C$$

$$= H_*(F_p C / F_{p-1} C)$$

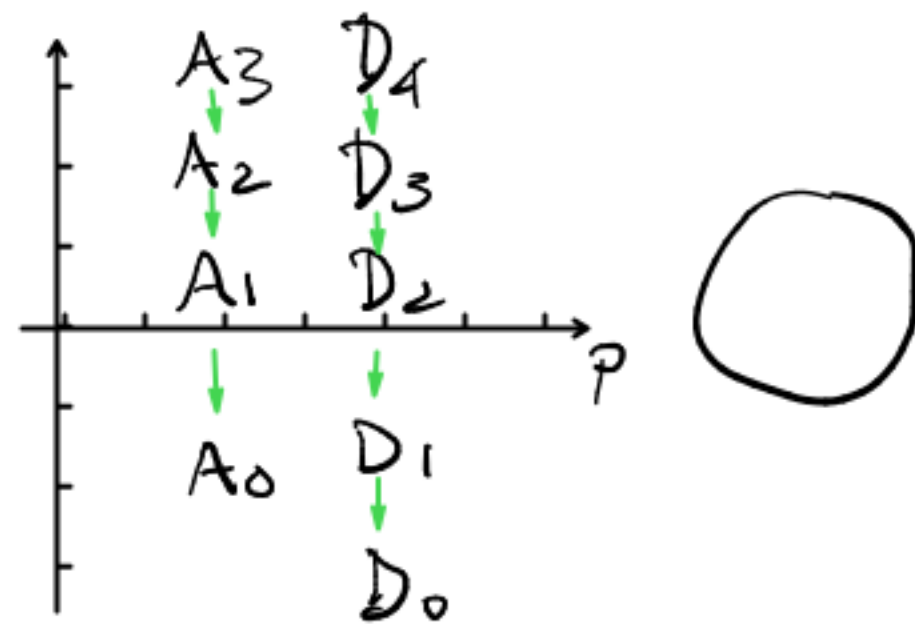
Proposition.  $\partial$  induces a differential on  $E^r$  of bidegree  $(-r, r-1)$  so that  $E^{r+1} = H(E^r)$ .

# Pre-class Warm-up!!

Starting from a short exact sequence of chain complexes  $0 \rightarrow A \rightarrow C \rightarrow D \rightarrow 0$ , what does the  $E^0$  page of the corresponding spectral sequence look like?



C



D Something else.

Each page of the s.s. comes with a differential  $d: E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$ .  
 $d$  has bidegree  $(-r, r-1)$ .  
 It is induced by the  $d$  on  $C$ .



Example. Consider a short exact sequence of chain complexes  $0 \rightarrow A \rightarrow C \rightarrow D \rightarrow 0$ .

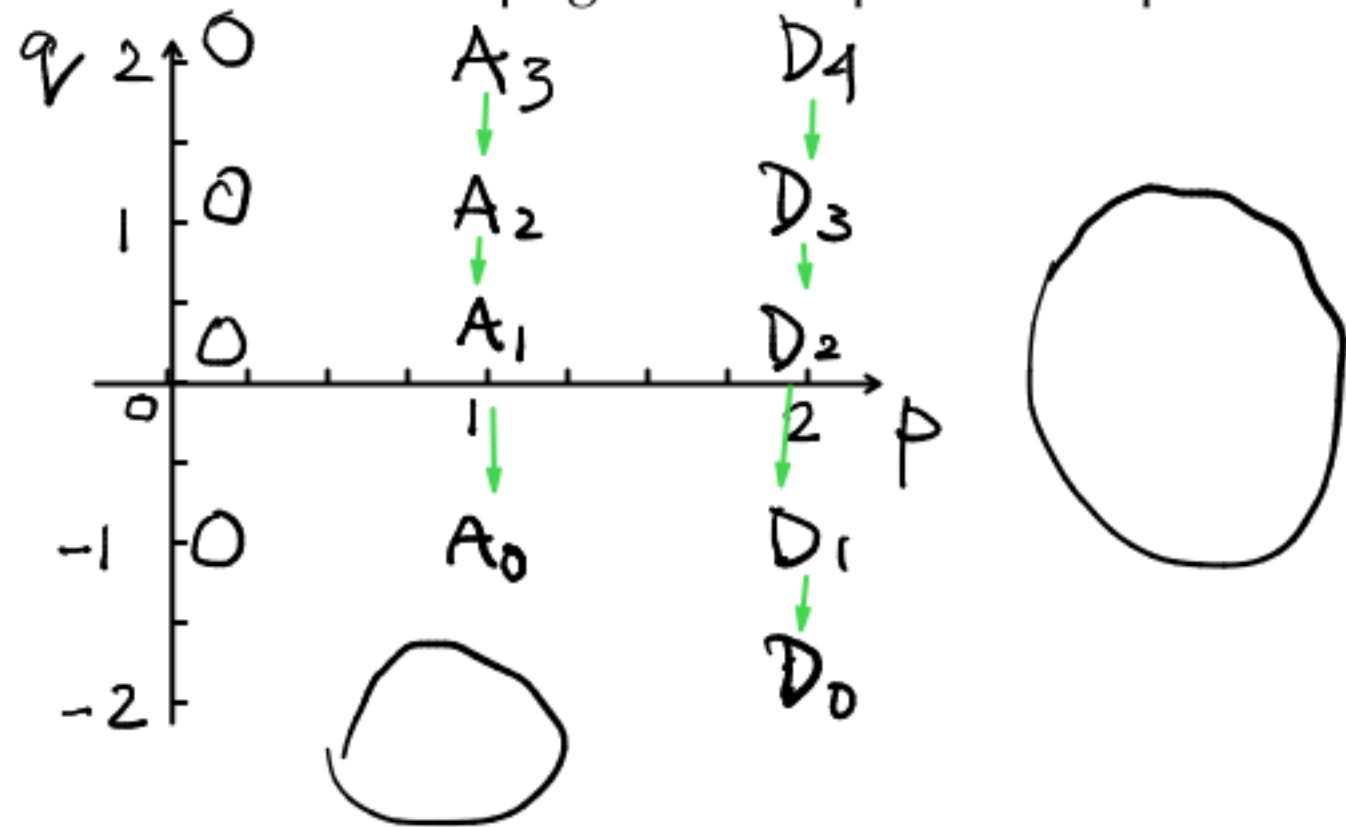
$$A_0 \leftarrow A_1 \leftarrow$$

This means we have a filtration of  $C$ .

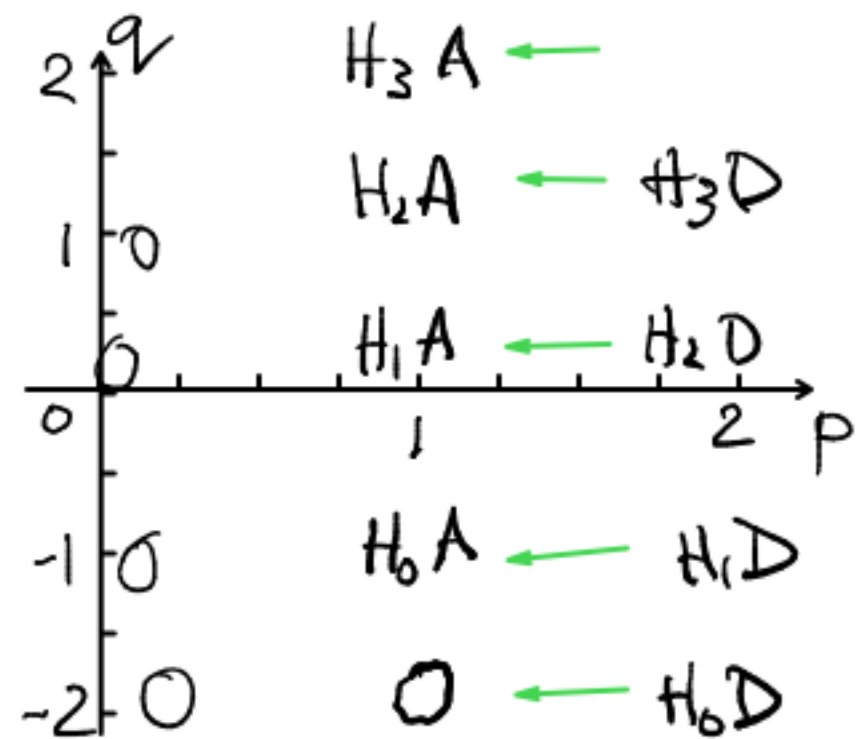
$$F_0 C = 0 \quad F_1 C = A \quad F_2 C = C.$$

$$\text{so } F_2 C / F_1 C = D.$$

We draw the  $E^0$  page of the spectral sequence



The  $E^1$  page.

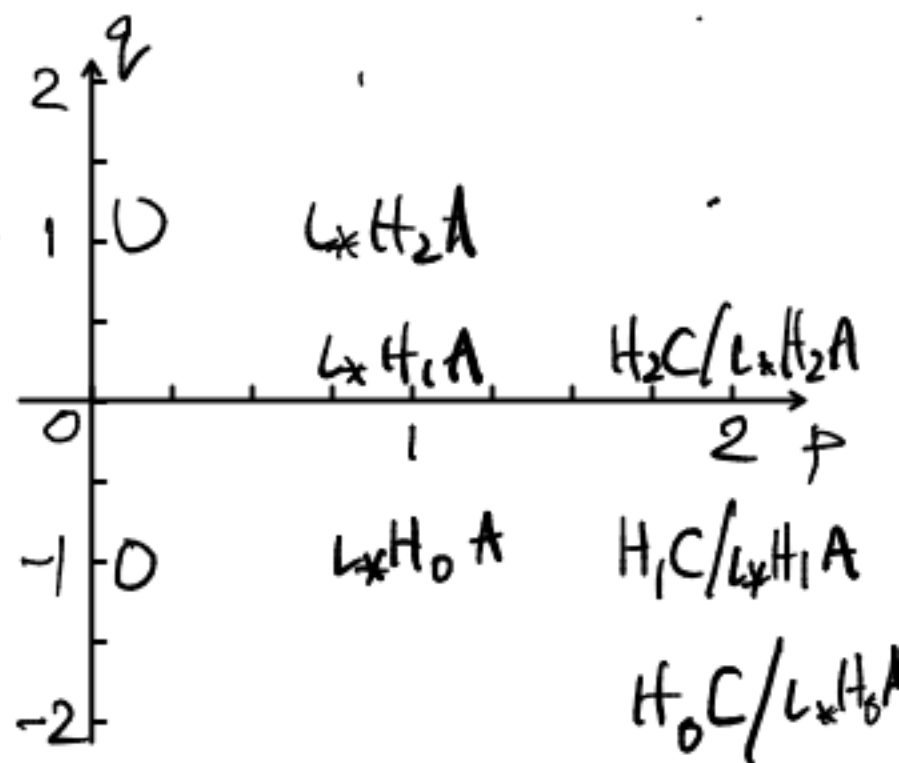


Fact  $d^1$  is the connecting homomorphism in the long e.s. in homology.

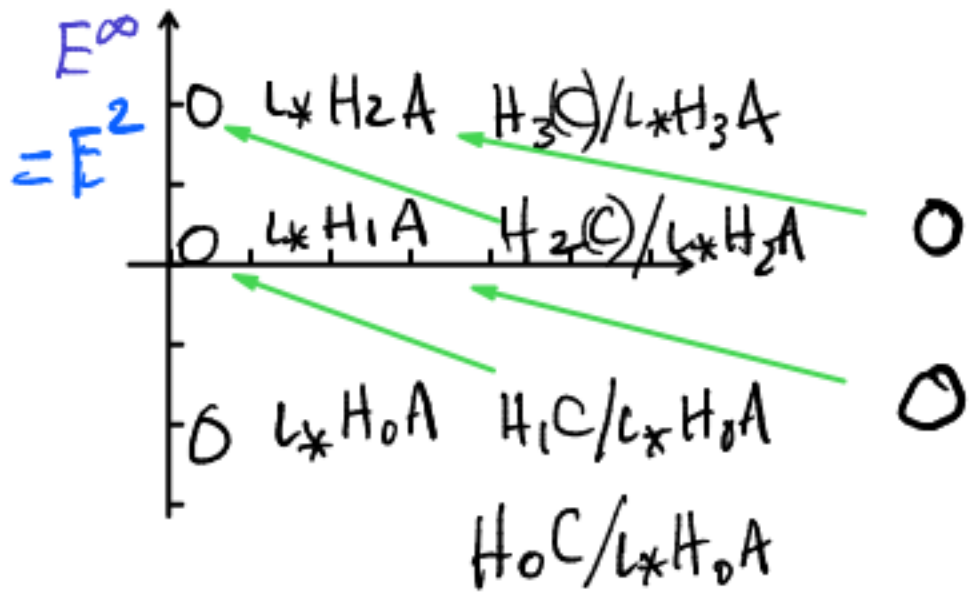
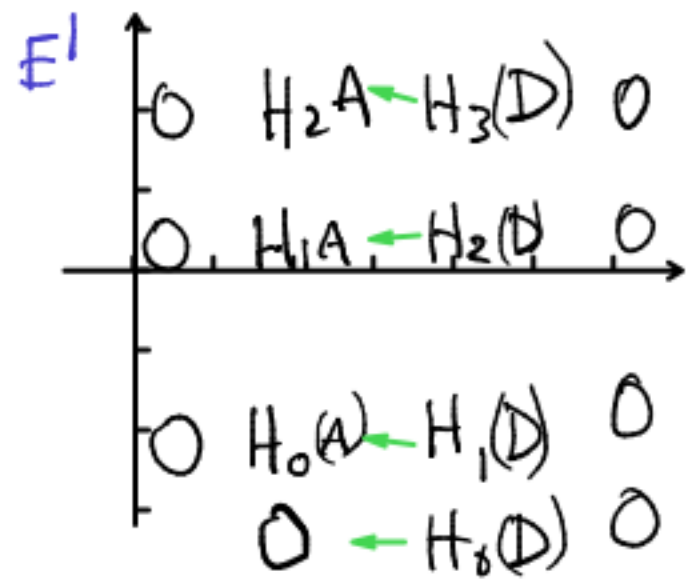
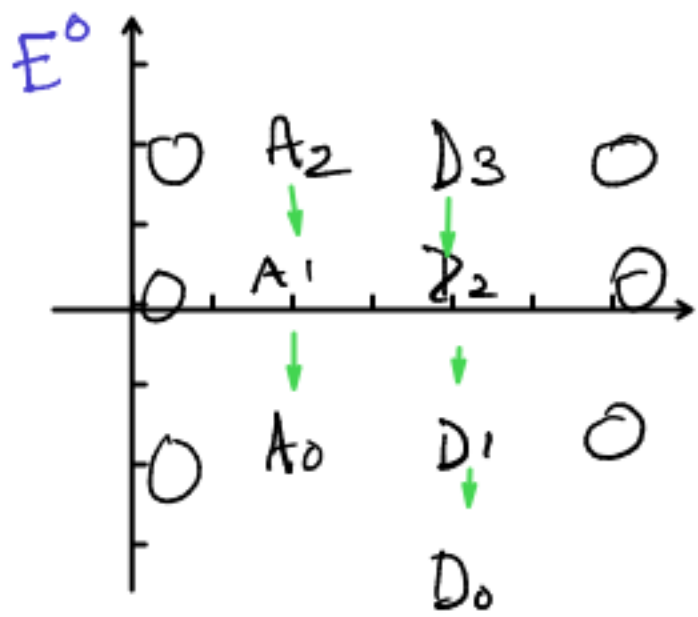
The  $E^\infty$  page.

$\cong$  the  $E^2$  page

because



The spectral sequence from  $0 \rightarrow A \rightarrow C \rightarrow D \rightarrow 0$



The long e.s. in  $H_*$  is

$$H_2 C \rightarrow H_2 D$$

$$H_1 A \xrightarrow{L_*} H_1 C \rightarrow H_1 D$$

$\rightarrow$

This gives exact sequences

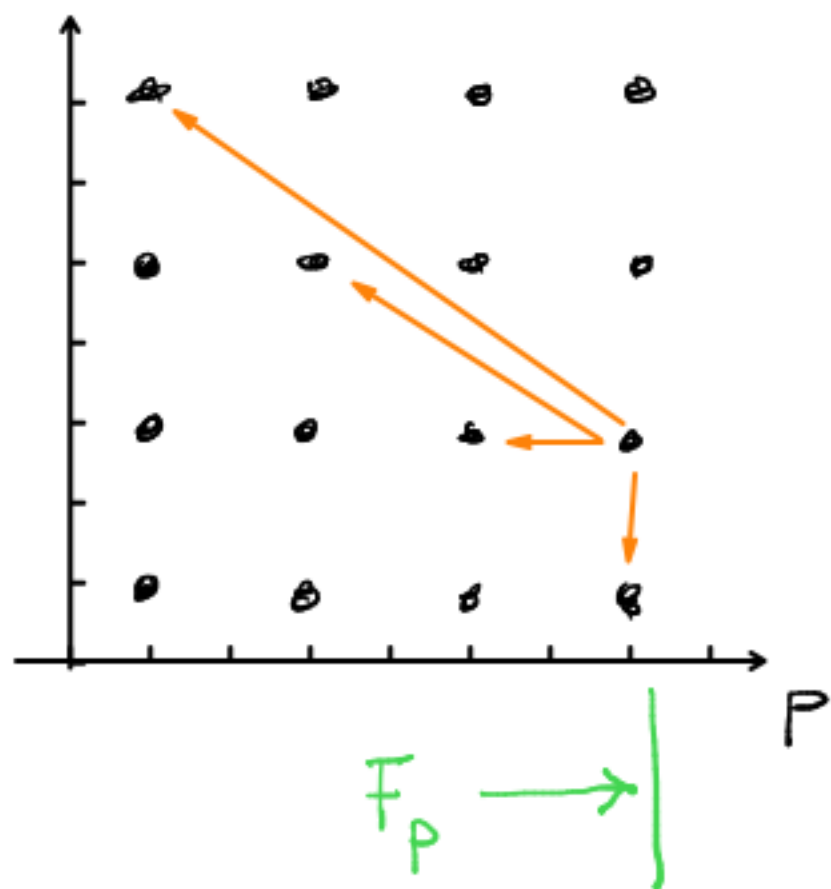
$$H_2 D \rightarrow H_1 A \rightarrow L_* H_1 A \rightarrow 0$$

$$0 \rightarrow H_2 C / L_* H_2 A \rightarrow H_2 D \rightarrow H_1 A$$

We see that the  $E^2$  page is the homology of the connecting homom. in the long e.s.

## The differentials on the spectral sequence

We recall the picture of the filtration of  $C$ :



At the  $E^r$  page we get the component of the full differential with bidegree  $(-r, r-1)$  only, because

the components to the right are zero on the homology of the differential on  $E^{r-1}$ ; the components to the left are 0 because we have factored out their boundary contributions.

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A spect. sequence with  $t$  columns have  $E^t = E^{t+1} = \dots = E^\infty$ .

2 columns has the  $E^\infty$  terms encoded in a long e.s.

Proposition Let  $\mu : C \rightarrow C'$  be a filtration-preserving chain map, where  $C$  and  $C'$  have degree-wise finite filtrations. If the induced map  $E^r(\mu) : E^r(C) \rightarrow E^r(C')$  of spectral sequences is an isomorphism for some  $r$ , then  $H(\mu) : H(C) \rightarrow H(C')$  is an isomorphism.

Spectral sequences can be used to compute Euler characteristics using any of their pages.

## Double complexes

### Definition

A double complex is a bigraded module  $C =$

With a 'horizontal' differential  $\partial'$  of bidegree  $(-1,0)$  and a 'vertical' differential  $\partial''$  of bidegree  $(0,-1)$  so that

The total complex  $TC$  has

Example. The tensor product of two chain complexes  $C'$  and  $C''$

## Example: the homology of a union.

Suppose that a simplicial complex  $X$  is the union of subcomplexes indexed by some totally ordered set  $J$ . We construct the nerve of this covering:

The  $p$ -simplices of  $K$

Let  $C_p$  be the chain complex