Spectral sequences

Source: I prefer the treatment in
K.S. Brown, Cohomology of groups,
chapter VII

Topics:
- the spectral sequence of a filtered complex
- how these arise from double complexes
- application to the homology of a union of spaces.

Motivation

We know that a short exact sequence of chain complexes 0 -> A. -> B. -> C. -> 0 gives rise to a long exact sequence in homology, perhaps giving information about H_*(B.)

Examples

1. Ext groups

Given a s.e.s. of R-modules $0 \to L \to M \to N \to 0$ we get a s.e.s of chain complexes

$$0 \to \mathrm{Hom}(P, L) \to \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N) \to 0$$

where $P \to A \to 0$ is a proj. resol^n of A, hence a long e.s.

$$0 \to \mathrm{Ext}^0_R(A, L) \to \qquad \to$$

$\to$ . . . etc.

2. We may have a simplicial complex $X \cup Y$ where $X \cap Y$ is a subsimplicial complex

We have a s.e.s. of chain cxes

$$0 \to C_\cdot(X \cap Y) \to C_\cdot(X) \oplus C_\cdot(Y) \to C_\cdot(X \cup Y) \to 0$$

Get long e.s. in homology.

What if the simplicial complex Δ has several subcomplexes

$X_1, \dots, X_n$.

$\Delta = \cup X_i$.

$C_\cdot(X_i) \subseteq C_\cdot(\Delta)$.

Let $F_p(\Delta)$ = span of the simplices in Δ that lie in at least p of the $X_1, \dots, X_n$.

We get subcomplexes

$$\cdots F_3(\Delta) \subseteq F_2(\Delta) \subseteq F_1(\Delta) \subseteq F_0(\Delta) = C_\cdot(\Delta)$$

Can we get info about

$H_*(C_\cdot(\Delta))$

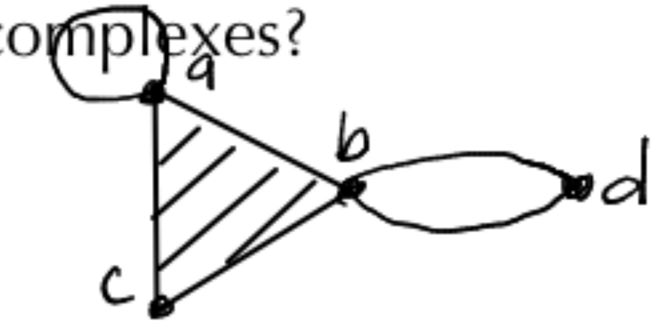
from the $H_*(F_p(\Delta)/F_{p+1}(\Delta))$?

Yes!

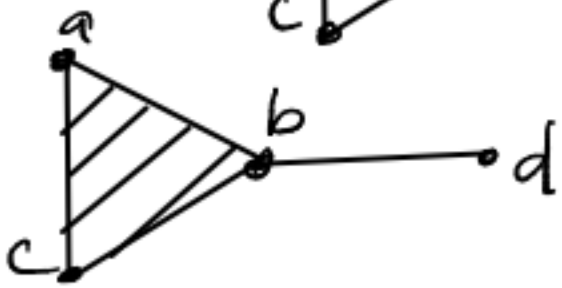
There is a spectral sequence generalizing the Mayer–Vietoris long e. s.

Pre-class Warm-up!!

Which of the following define the same simplicial complexes?

1. 

Not a simplicial complex

2. 

3. $\{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{a,b,c\}\}$

A 1 and 2 describe the same simplicial complex.

B 1 and 3 describe the same simplicial complex.

C 2 and 3 describe the same simplicial complex.

D They all describe the same simplicial complex.

An (abstract) simplicial complex is a set Δ of subsets of a set S so that $T \in \Delta$, $U \subseteq T$ $\Rightarrow$ $U \in \Delta$.

Filtrations of modules and associated graded modules

An ascending filtration of a module M is a chain of submodules

$$\cdots \subseteq F_p(M) \subseteq F_{p+1} \subseteq \cdots \subseteq M.$$

A $\mathbb{Z}$ graded module is a list of modules M_p, $p \in \mathbb{Z}$.
We may want to think of it as $\bigoplus_{p \in \mathbb{Z}} M_p$.

Given a filtration the associated graded module $\operatorname{Gr} M$ has

$$\operatorname{Gr}_p M = F_p M / F_{p-1} M.$$

e.g. $k[x] = \bigoplus_{p \geq 0} k x^p$

We assume that filtrations are finite.

This means $F_p = F_{p+1} = \cdots$ if p is large enough. and $F_p = F_{p-1} = \cdots$ if p is small enough.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

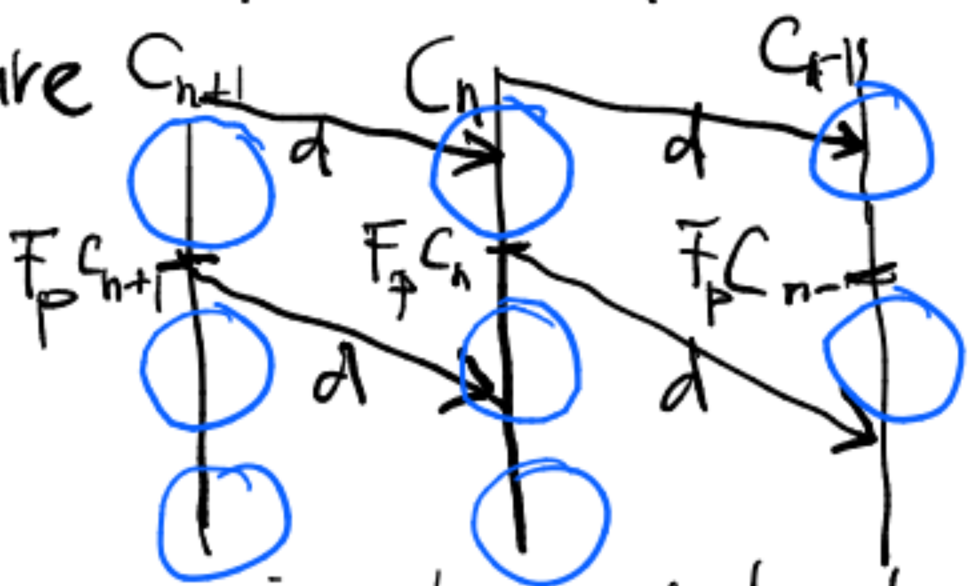
D Shaky

Definition.

A filtration of a chain complex $C_\cdot$ is a chain (list) of subcomplexes

$$\cdots \subseteq F_p(C_\cdot) \subseteq F_{p+1}(C_\cdot) \subseteq \cdots$$

Picture

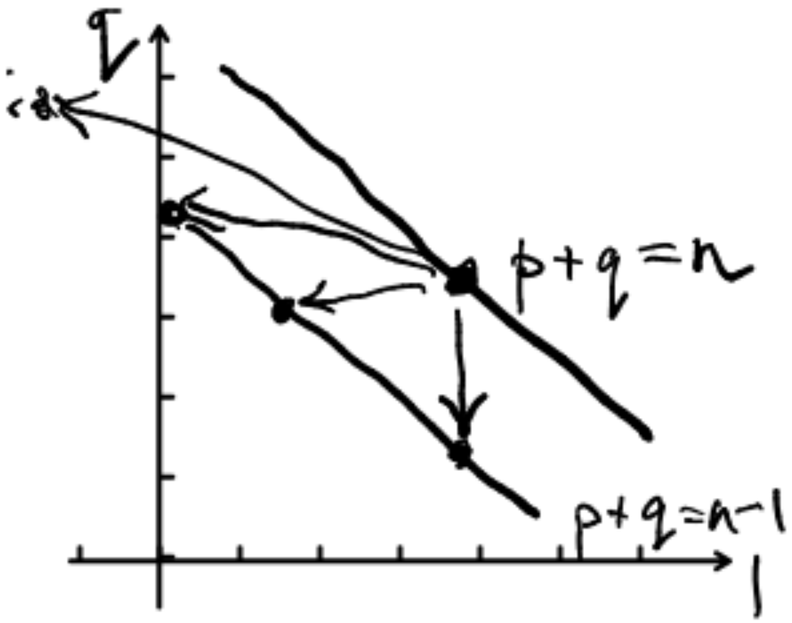


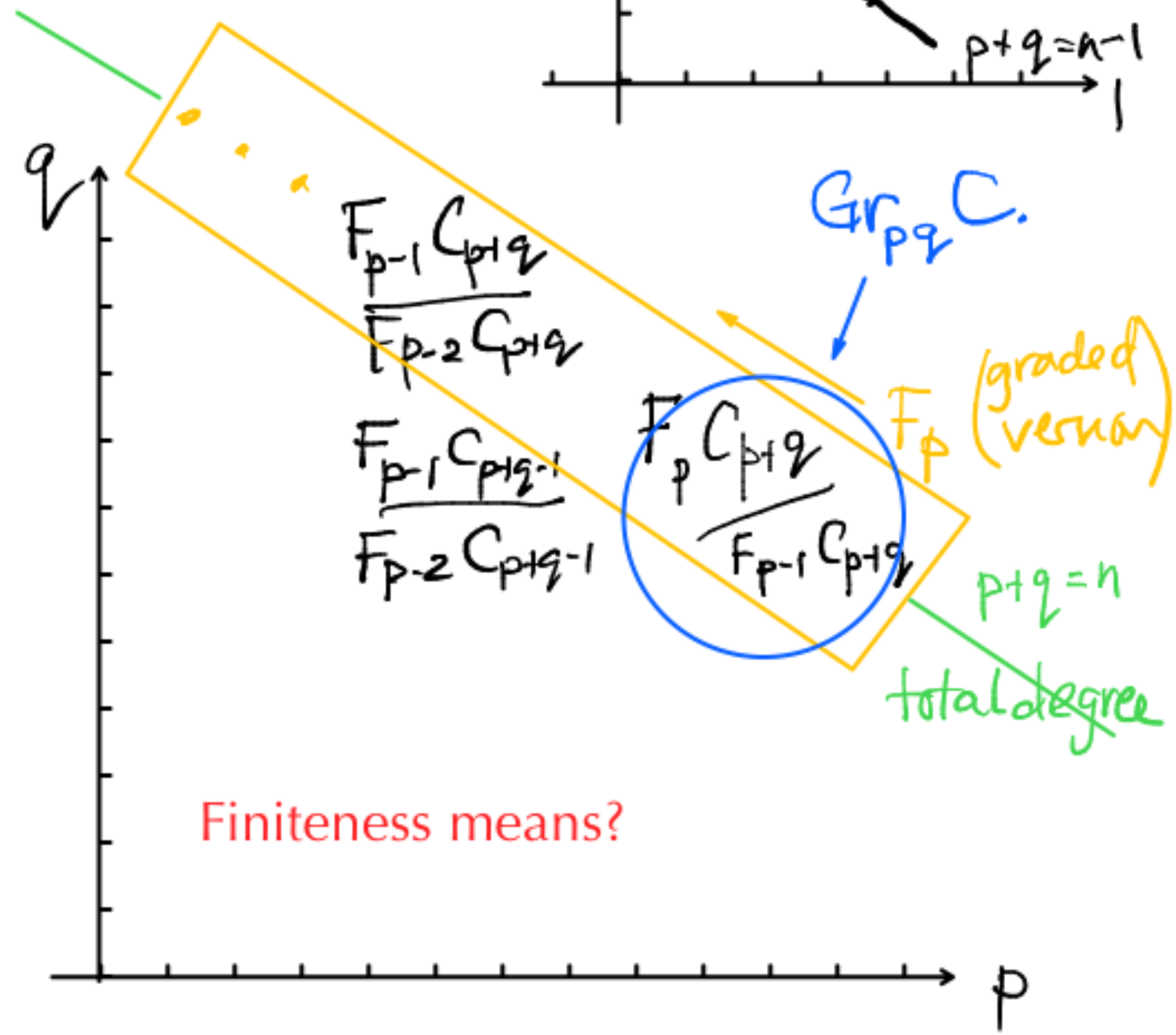
The associated graded object has $Gr_p C_\cdot = F_pC_\cdot / F_{p-1}C_\cdot$

which is also a list of modules indexed by homological degree n.

We consider filtrations finite in each homological degree.

The differential looks like





Finiteness means?

Pre-class Warm-up!!

Suppose we have a chain complex C. That is filtered

$$\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots$$

Writing the terms of the associated graded complex on a grid, as we did last time, where would we position the term of F_5 C / F_4 C that is in homological degree 7?

A At position (5,7)

B At position (4,7)

C At position (5,2)

D At position (7,5)

E None of the above.

homological degree of E_{pq} is $p+q$

p indexes the filtration.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

D Shaky

The filtration on the homology of a filtered complex.

Set up: we have a filtration $F_pC \subseteq F_{p+1}C \subseteq \cdots \subseteq C$. The inclusion $F_pC \xrightarrow{\iota} C$ gives a map in homology $H_*(F_p(C)) \xrightarrow{\iota_*} H_*(C)$.

Define $F_p(H_*(C)) =$ Image of ι_*.

This gives a filtration of $H_*(C)$ and an associated graded group.

Write $H_*(C) = Z/B$ suppressing homological degree

Proposition.

a. The image of H.(F_p C) in H.(C) is

$$(F_pC \cap Z)/(F_pC \cap B)$$

$$Gr_p H(C) = (F_pC \cap Z)/((F_pC \cap B) + (F_{p-1}C \cap Z))$$

Proof a. $H_*(F_pC) = \dfrac{\text{cycles of } F_pC}{\partial(F_pC)}$

$$= \frac{F_pC \cap Z}{\partial(F_pC)}$$

The map $H_*(F_pC) \to H_*C$ is induced by the $F_pC \cap Z \to Z/B$ and surjects to $F_p(H_*C)$.

The kernel is $(F_pC \cap Z) \cap B = F_pC \cap B$

b. $Gr_p(HC) = F_p(HC)/F_{p-1}(HC)$

$$= \frac{(F_pC \cap Z)+B}{B} \Big/ \frac{(F_{p-1}C \cap Z)+B}{B}$$

$$= \frac{(F_pC \cap Z)+B}{(F_{p-1}C \cap Z)+B} \cong \frac{F_pC \cap Z}{((F_{p-1}C \cap Z)+B) \cap (F_pC \cap Z)}$$

$$= \frac{F_pC \cap Z}{(F_{p-1}C \cap Z) + (B \cap F_pC)}$$ by the modular law

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

D Shaky

The spectral sequence of a filtered complex

We define for each $r = 0, 1, 2, \ldots$

$$Z^r_{pq} = F_pC_{p+q} \cap \partial^{-1} F_{p-r} C_{p+q-1}$$

$\partial : C_{p+q} \to C_{p+q-1}$

$$Z^\infty_p = F_pC \cap Z$$

$$B^r_{pq} = F_pC_{p+q} \cap \partial F_{p+r-1} C_{p+q+1}$$

$$B^\infty_p = F_pC \cap B$$

Pre-class Warm-up!
Study this setup and get familiar with it.

Proposition.
Assume the filtration is finite in each homological degree. Then

$$B^0_p \subseteq B^1_p \subseteq \cdots \subseteq B^\infty_p \subseteq Z^\infty_p \subseteq \cdots$$
$$\subseteq Z^1_p \subseteq Z^0_p = F_pC$$

In each degree the B and Z sequences stabilize.

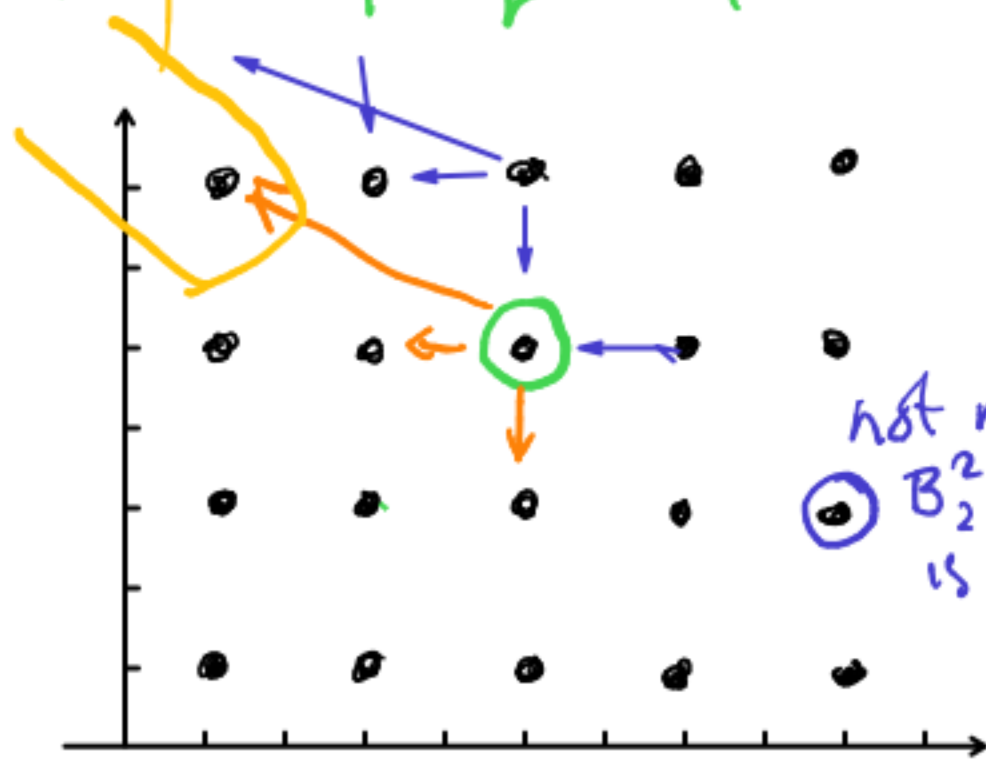
$Z^2_{2,2}$ = all elts of F_2C_4 that map into F_0C_3

$$B^2_{22} = \partial F_3 C_5$$

Proof B^r_p is the image of something bigger than B^{r-1}_p is. As r increases Z^r_p is the preimage of something smaller.

$F_{p-r}C_{p+q-1} = F_0C_3$

$r = 2 \quad p+q=4 \quad p=2 \quad \partial$



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not mapped to B^2_{22} b/c it is outside F_3C.

How did that work for you?

A I so totally got that

B OK

C I’m not sure about what we just did.

D Shaky

Definition

$$E^r_{pq} = Z^r_{pq}/(B^r_{pq} + Z^{r-1}_{p-1,q})$$

$$= Z^r_p/(B^r_p + (F_{p-1}C \cap Z^r_p))$$

$$E^\infty_{pq} = Z^\infty_{pq}/(B^\infty_{pq} + Z^\infty_{p-1,q}) = \mathrm{Gr}_p\, H(C)_{p+q}$$

Proposition.

a. $Z^{r-1}_{p-1} = F_{p-1}C \cap Z^r_p$

b. $Z^\infty_p/(B^\infty_p + Z^\infty_{p-1}) = \mathrm{Gr}_p\, H(C)$

c. For fixed (p,q) we have

$$E^r_{pq} = E^{r+1}_{pq} = \ldots = E^\infty_{pq}$$

For r sufficiently large. The sequence 'converges' to Gr H(C) as $r \to \infty$.

Which seems hardest? A a. B b. C c.

Proof a.

$Z^{r-1}_{p-1} = F_{p-1}C \cap Z^r_p$ b/c

Z^r_p = those x in F_pC, $\partial x \in F_{p-r}C$

Z^{r-1}_{p-1} = those x in $F_{p-1}C$, $\partial x \in F_{p-r}C$

b. $\mathrm{Gr}_p\, H(C) = \underbrace{(F_pC \cap Z)}_{Z^\infty_p} / [\underbrace{(F_pC \cap B)}_{B^\infty_p} + \underbrace{(F_{p-1}C \cap Z)}_{Z^\infty_{p-1}}]$

c. The terms $B^r_p \subseteq B^{r+1}_p \subseteq B^\infty_p$ stabilize with r. So do the Z^r_p.

Assume F_pC stabilizes in each homological degree at C Then we get $= E^\infty_{pq}$

Other terminology: H(C) is the abutment of the spectral sequence.

How did that work for you?

A I so totally got that

B OK

C I'm not sure about what we just did.

D Shaky

Question:

$E_p^r = Z_p^r / B_p^r$ and $E_p^\infty = Z_p^\infty / B_p^\infty$?

Proposition.
The E^0 and E^1 pages of the spectral sequence are as follows:

a. $E^0_p = F_pC / F_{p-1}C = Gr_p C$

b. $E^1_p = H_\cdot(F_pC / F_{p-1}C)$

Thus E^1 is the homology of E^0, relative to the differential induced on E^0 by ∂

Proof. a. $E^0_p = Z^0_p / (B^0_p + Z^{-1}_{p-1})$
We take $Z^{-1}_{p-1} = Z^0_{p-1} \doteq F_{p-1}C$
$B^0_p = F_pC \cap \partial F_{p-1}C \subseteq F_{p-1}C$.
$E^0_p = F_pC / F_{p-1}C$
b. $E^1_p = Z^1_p / (B^1_p + Z^0_{p-1})$
$= (F_pC \cap \partial^{-1}F_{p-1}C) / ((\partial F_pC \cap F_pC) + F_{p-1}C)$

Where would you draw E_p^0 on the grid?

A the vertical line distance p from the origin.

B the horizontal line distance p from the origin

C the slope -1 line distance p from the origin.

D at coordinate (p,0)

$=$ things sent to 0 in $F_pC/F_{p-1}C$ by ∂ / Image of $\partial : F_pC/F_{p-1}C \to F_pC/F_{p-1}C$

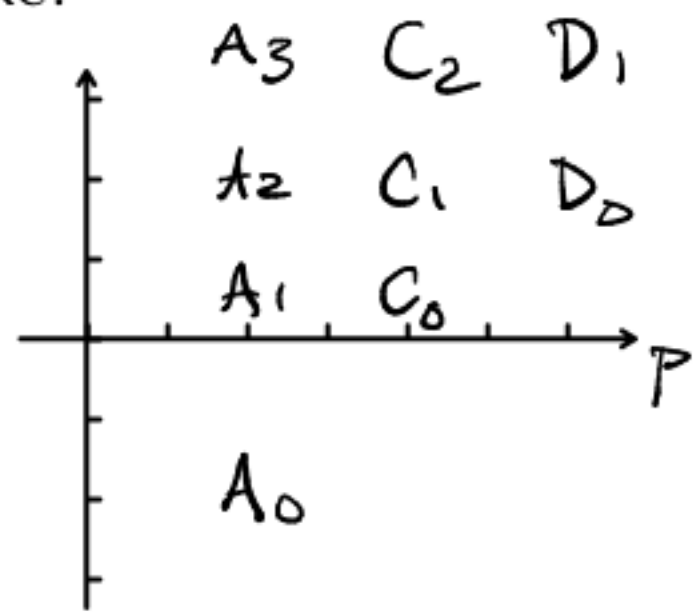
$= H_\cdot(F_pC / F_{p-1}C)$

Proposition. ∂ induces a differential on E^r of bidegree (-r, r-1) so that E^{r+1} = H(E^r).

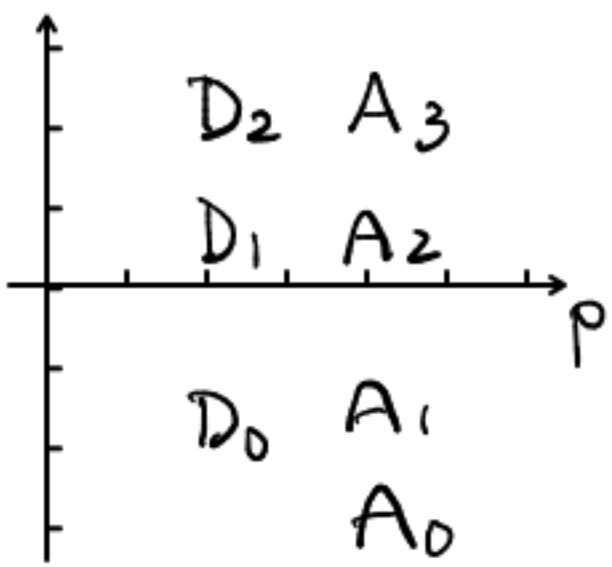
Pre-class Warm-up!!

Starting from a short exact sequence of chain complexes 0 -> A -> C -> D -> 0, what does the E^0 page of the corresponding spectral sequence look like?

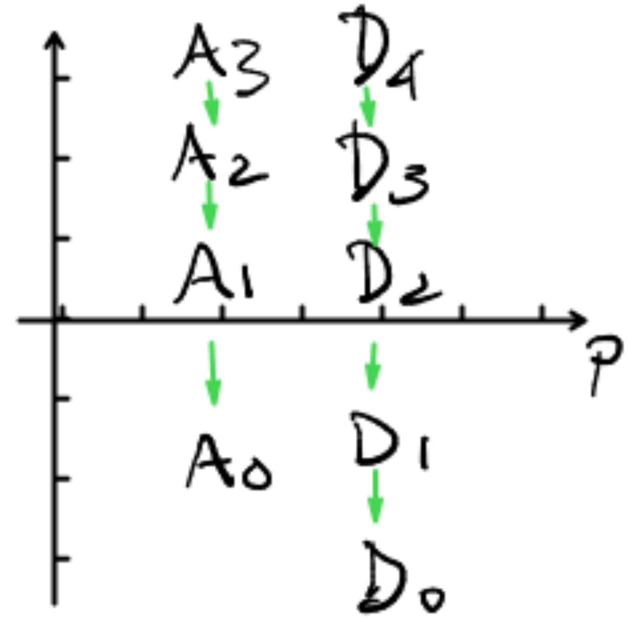
A



B



C



D Something else.

Each page of the s.s. comes with a differential

$$d: E^r_{p,q} \longrightarrow E^r_{p-r,\,q+r-1}$$

d has bidegree $(-r, r-1)$.

It is induced by the d on C.

Example. Consider a short exact sequence of chain complexes 0 -> A -> C -> D -> 0.

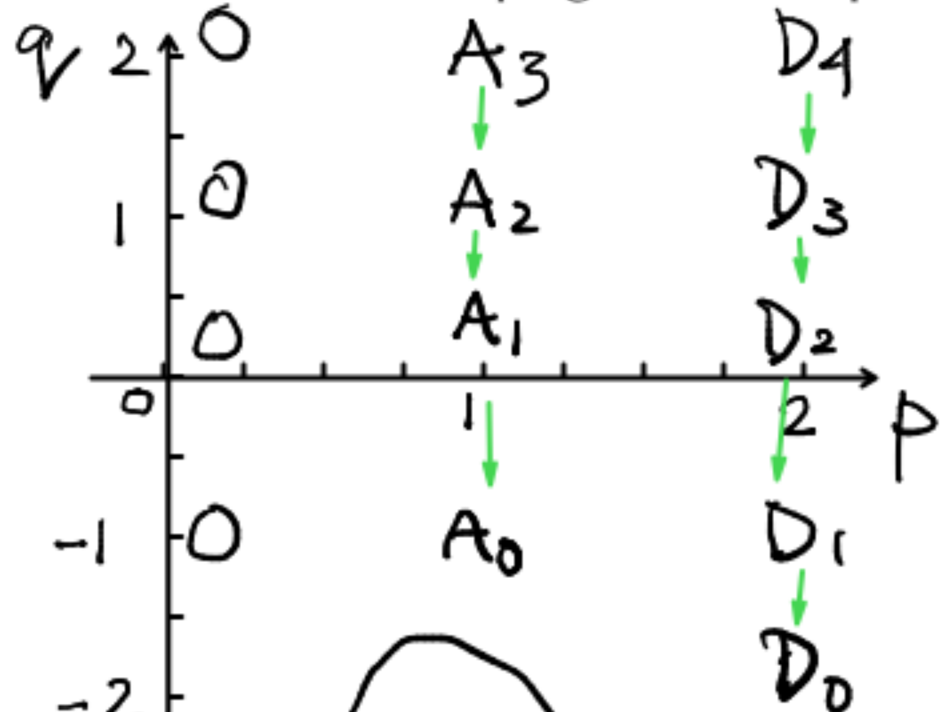
$A_0 \xleftarrow{\partial} A_1 \leftarrow$

This means we have a filtration of C.

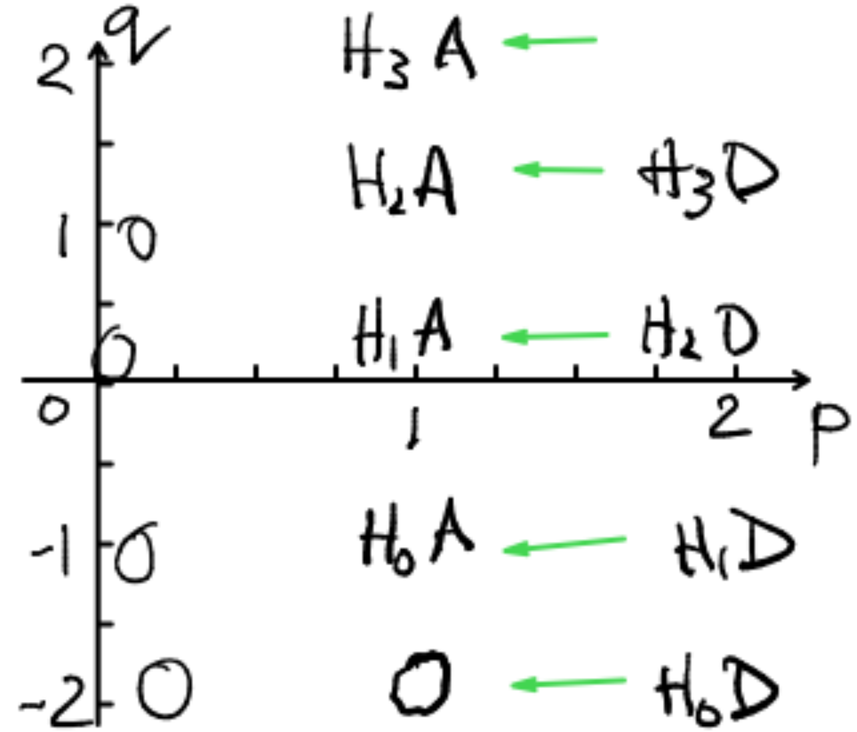
$F_0C = 0 \qquad F_1C = A. \qquad F_2C = C.$

so $F_2C / F_1C = D.$

We draw the E^0 page of the spectral sequence



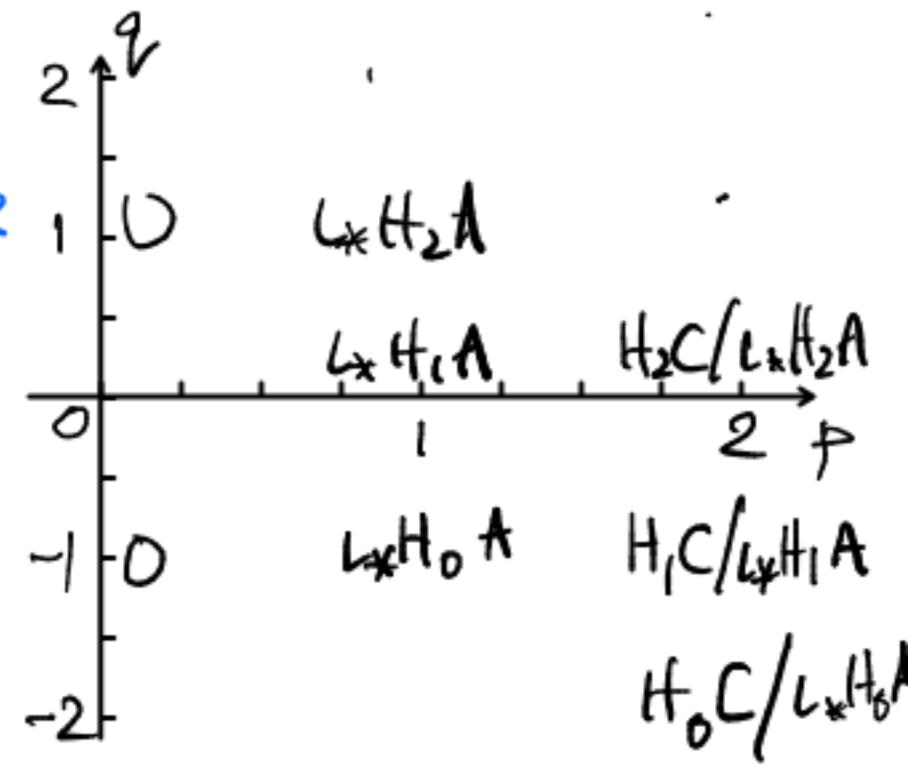
The E^1 page.



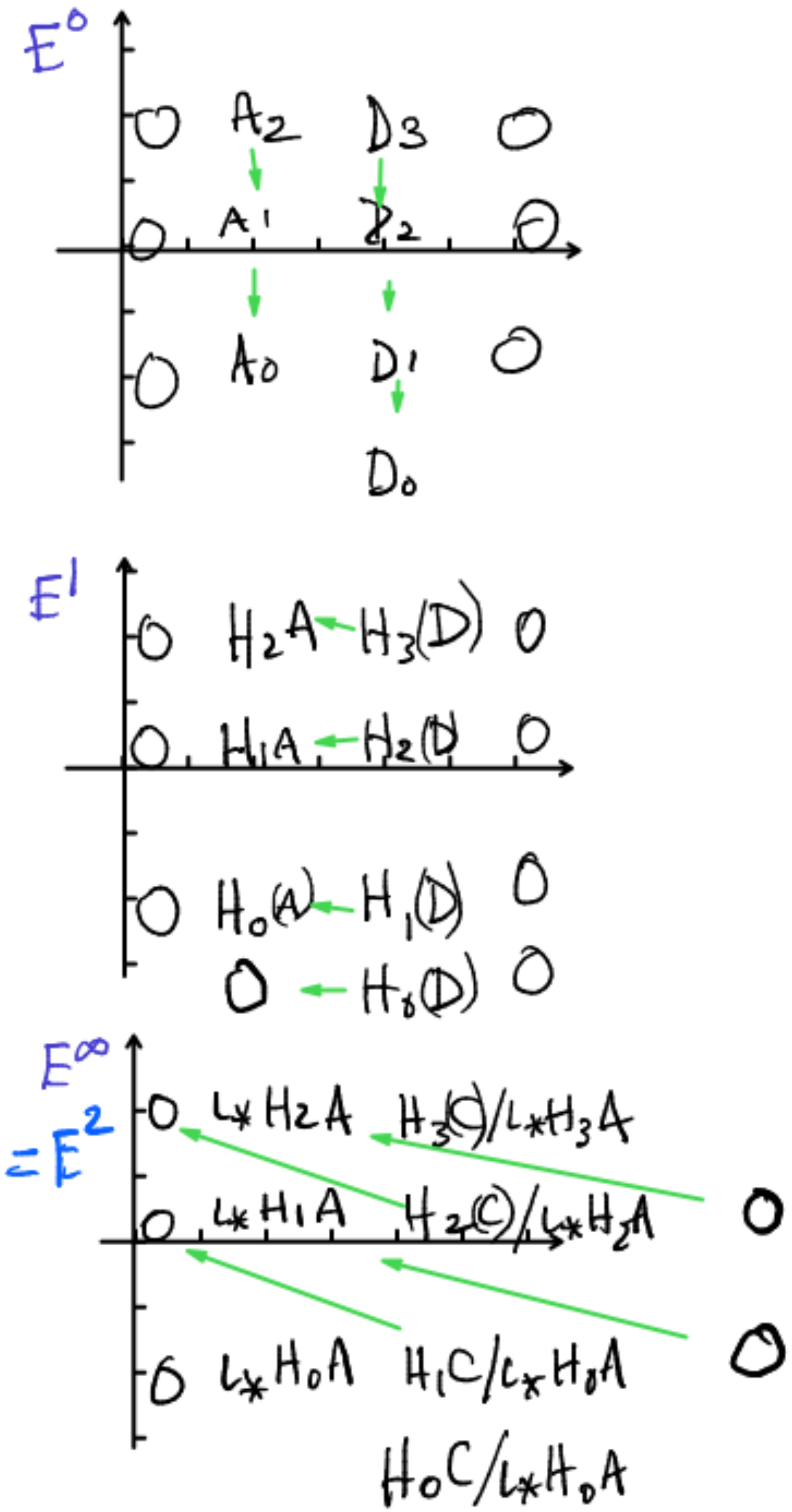
Fact d^1 is the connecting homom in the long e.s. in homology.

The E^∞ page.

= the E^2 page because



The spectral sequence from 0 -> A -> C -> D -> 0



The long e.s. in H_* is

$$H_2 C \to H_2 D \to H_1 A \xrightarrow{i_*} H_1 C \to H_1 D \to$$

This gives exact sequences

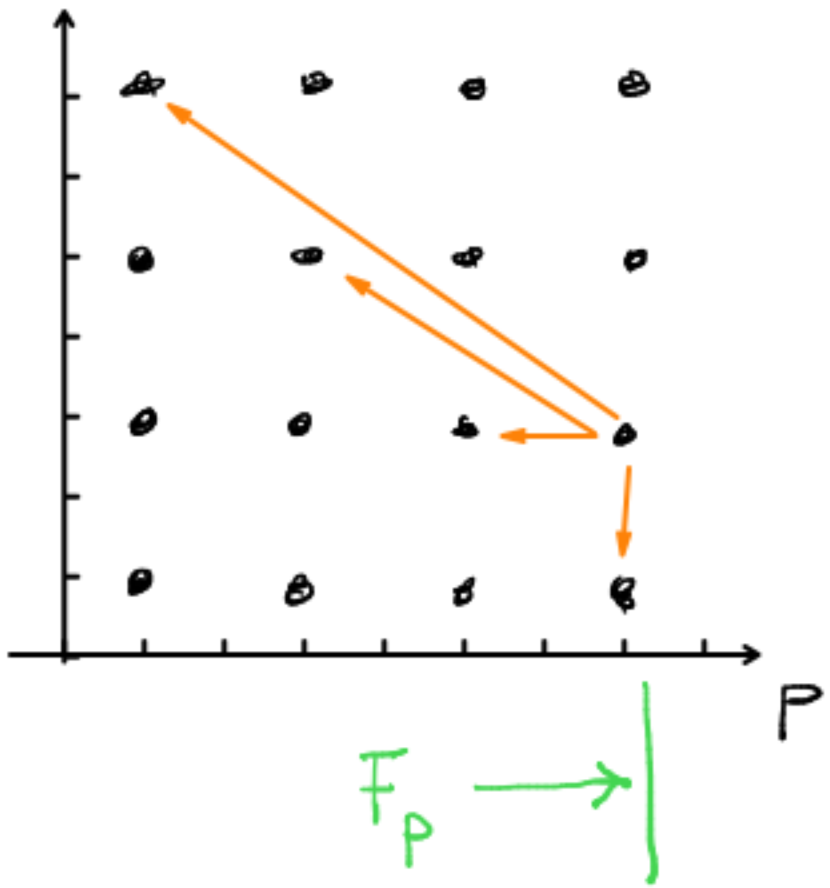
$$H_2 D \to H_1 A \to i_* H_1 A \to 0$$

$$0 \to H_2 C / i_* H_2 A \to H_2 D \to H_1 A.$$

We see that the E^2 page is the homology of the connecting homom. in the long e.s.

The differentials on the spectral sequence

We recall the picture of the filtration of C:



At the E^r page we get the component of the full differential with bidegree $(-r, r-1)$ only, because the components to the right are zero on the homology of the differential on E^{r-1}; the components to the left are 0 because we have factored out their boundary contributions.

A spect. sequence with t columns have $E^t = E^{t+1} = \ldots = E^\infty$.

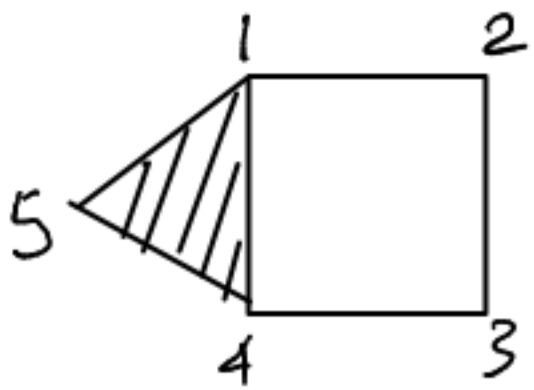
2 columns has the E^∞ terms encoded in a long e.s.

Proposition Let $\mu : C \to C'$ be a filtration-preserving chain map, where C and C' have degree-wise finite filtrations. If the induced map $E^r(\mu) : E^r(C) \to E^r(C')$ of spectral sequences is an isomorphism for some r, then $H(\mu) : H(C) \to H(C')$ is an isomorphism.

Spectral sequences can be used to compute Euler characteristics using any of their pages.

Pre-class Warm-up!!!

Wthat is the rank of the degree 1 term C_1(X) in the simplicial chain complex of the simplicial complex

$X =$ 

A   1

B   3

C   5

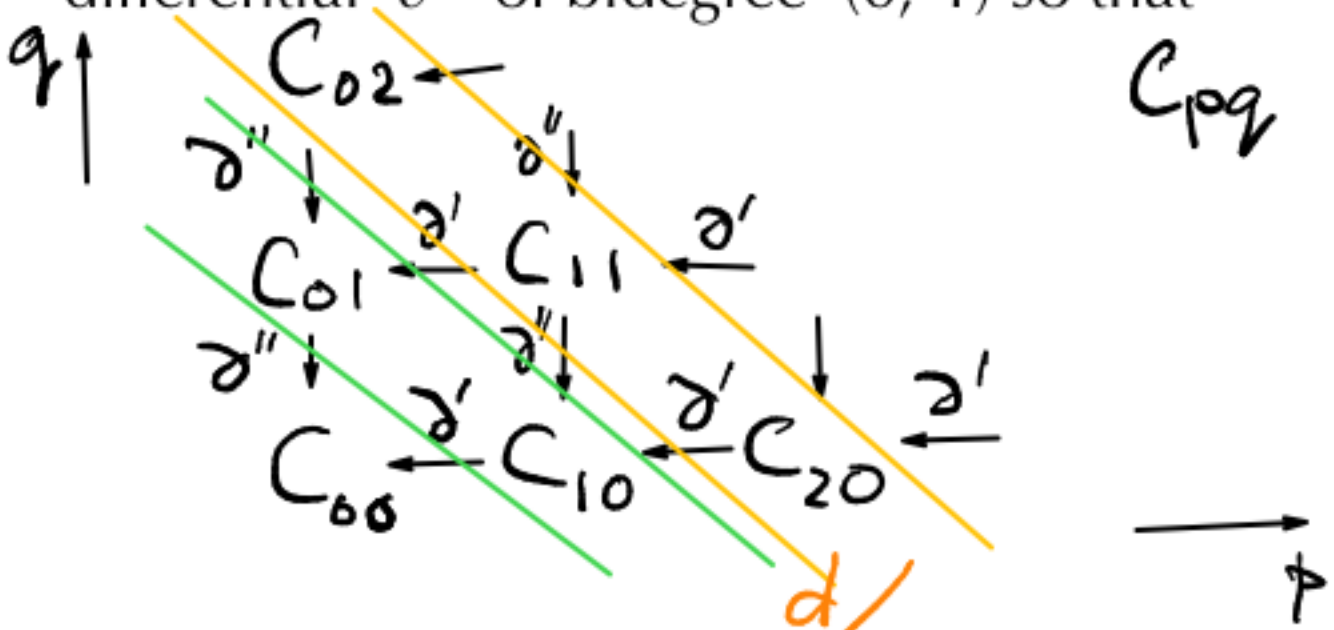
D   6

$C_t(X) :=$ free R-module with basis the simplices in dimension t.

Double complexes

Definition
A double complex is a bigraded module $C = C_{pq}$, $p,q \in \mathbb{Z}$

With a 'horizontal' differential ∂' of bidegree (-1,0) and a 'vertical' differential ∂'' of bidegree (0,-1) so that



so that all squares commute:

$\partial'\partial'' = \partial''\partial'$ always

and where $\partial'\partial' = 0$, $\partial''\partial'' = 0$

We can regard a double complex as a chain complex in the category of chain complexes – in 2 ways!

The total complex TC has

$$TC_n = \bigoplus_{p+q=n} C_{pq} \quad \text{with}$$

boundary map $d : TC_n \to TC_{n-1}$

with components $d|_{C_{pq}} = \partial'_{pq} + (-1)^p \partial''$

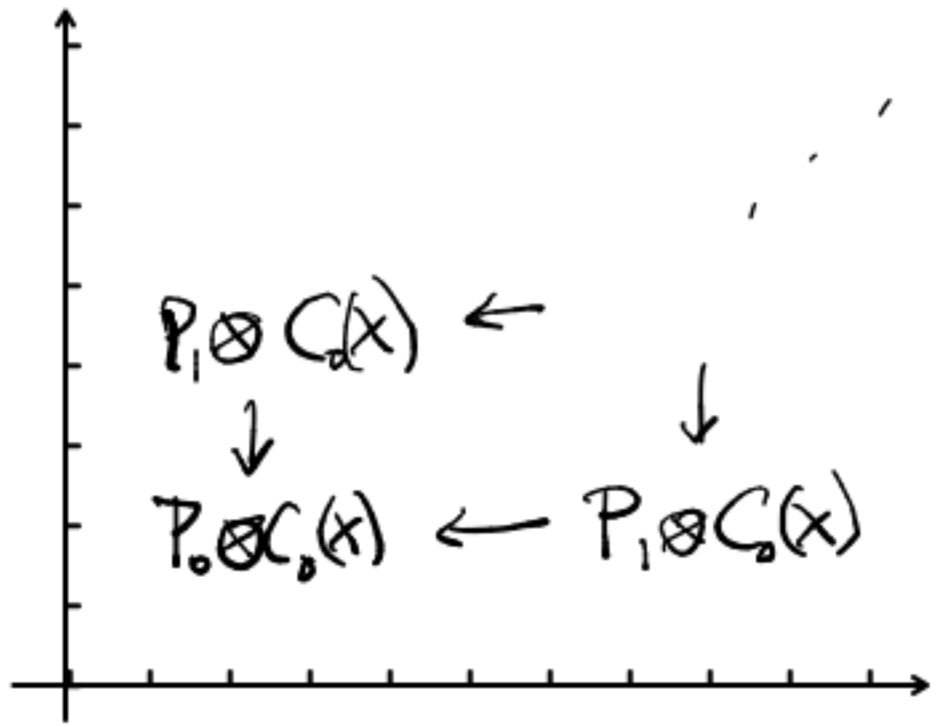
so that $dd = 0$.

Example. The tensor product of two chain complexes C' and C'' over a commutative R.

$$C' = \leftarrow C'_{p-1} \leftarrow C'_p \leftarrow \cdots$$

$$C'' = C''_{p-1} \leftarrow C''_p \leftarrow \cdots$$

$$\begin{tikzcd}
C'_{p-1} \otimes C_q \arrow[d] & C'_p \otimes C''_q \arrow[l, "\partial' \otimes 1"'] \arrow[d, "1 \otimes \partial''"] \\
C'_{p-1} \otimes C''_{q-1} & C'_p \otimes C''_{q-1} \arrow[l]
\end{tikzcd}$$



Example. G is a finite group acting simplicially on a simplicial complex X. The chain complex

$$C(X) = \quad \leftarrow C_p(X) \leftarrow C_{p+1}(X)$$

is a complex of $\mathbb{Z}G$-modules.

We may take a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$\mathcal{P} = \left(P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \right)$$

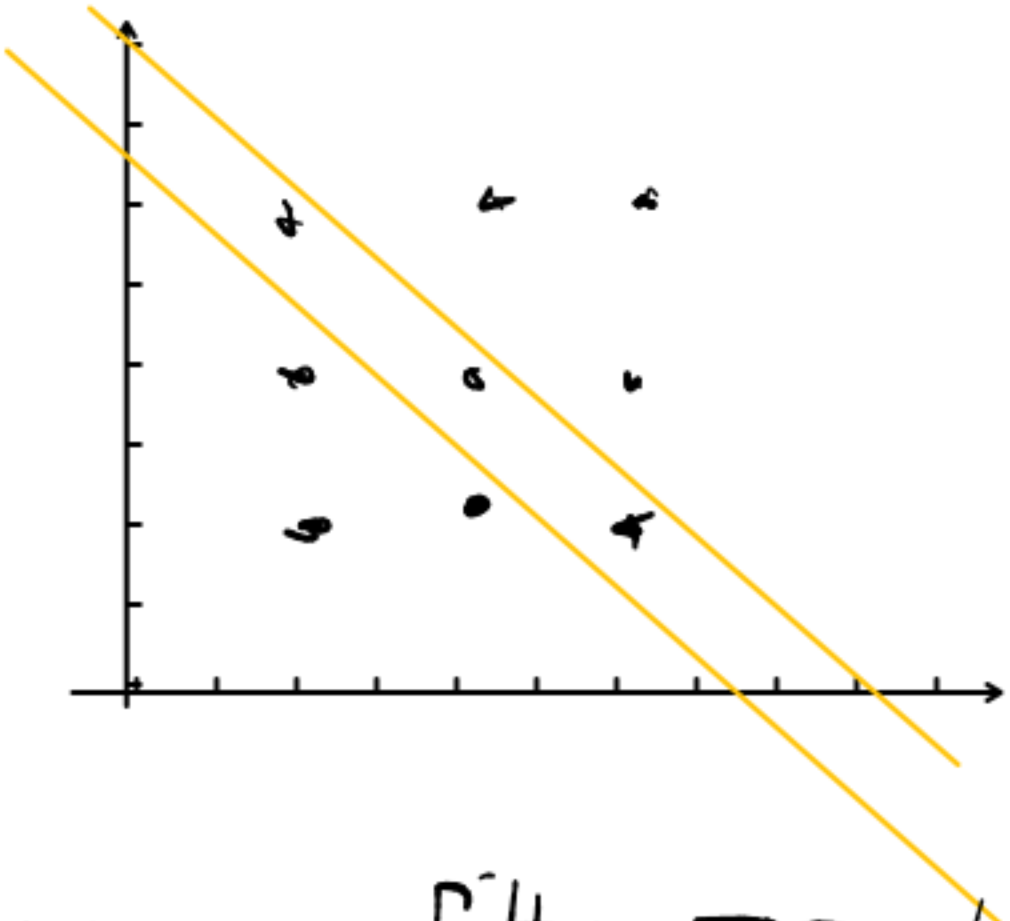
$$P_0 \to \mathbb{Z} \to 0$$

Form the double complex

$$\mathcal{P} \otimes_{\mathbb{Z}G} C(X).$$

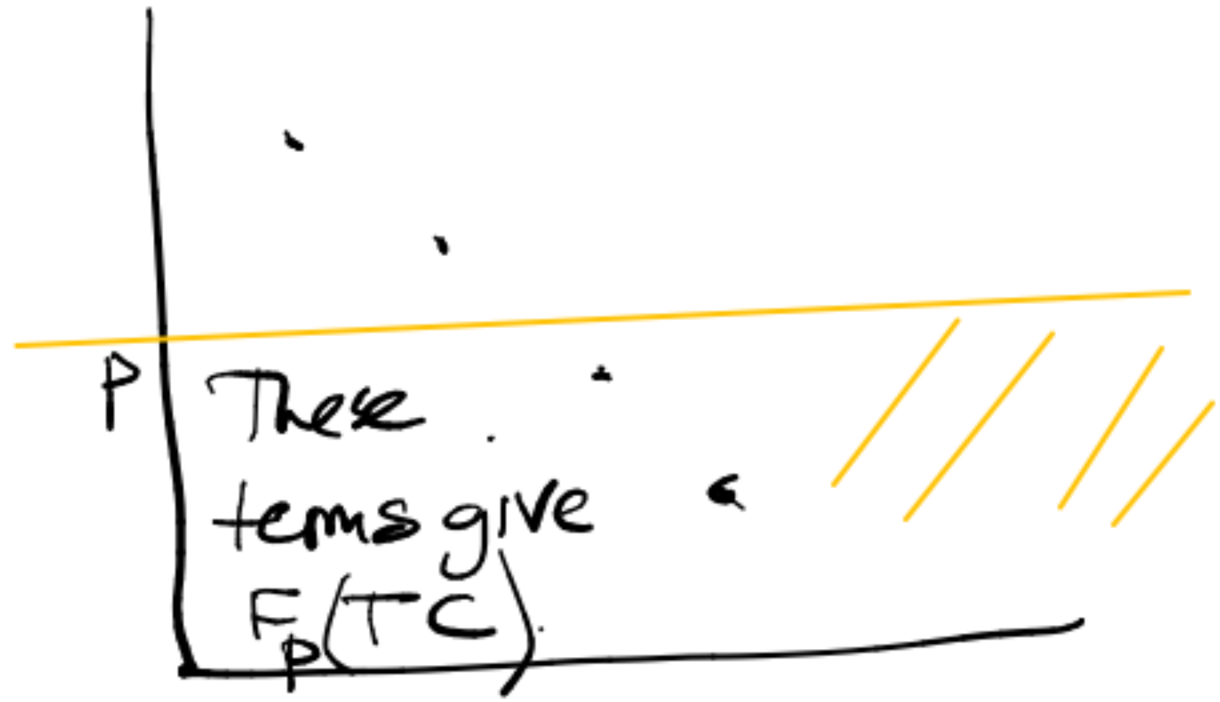
Definition: The homology of $T(\mathcal{P} \otimes_{\mathbb{Z}G} C(X))$ is equivariant homology of G acting on X: $H_*(G;X)$.

The spectral sequences associated to a double complex:



We can filter TC by columns: $F_p TC = \bigoplus$ terms to the left of

We can also filter using rows



If we have a 1st quadrant spectral sequence, the filtrations are finite in each homological degree. The spectral sequences both converge to $H_*(TC)$

Pre-class Warm-up!!

Suppose we have a chain complex in the category of chain complexes. Do we understand what its homology is, say in degree 5? Is it best described as

A a module

B a graded module

C a chain complex of modules ✓

D a chain complex of chain complexes

E None of the above

Example: the homology of a union.

Suppose that a simplicial complex X is the union of subcomplexes X_i indexed by some totally ordered set J. We construct the nerve of this covering: $i \in J$

It is a simplicial complex K with vertex set J

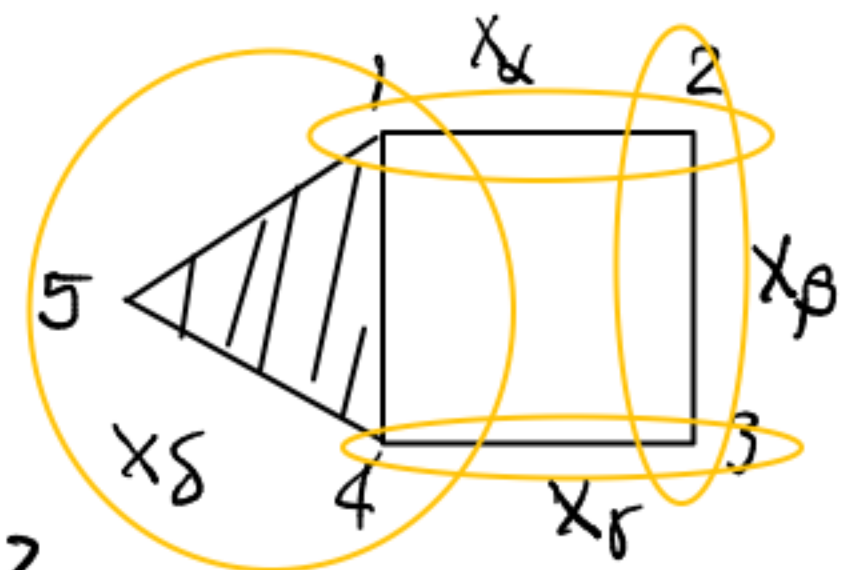
The p-simplices of K are

Put an order on J. $\alpha < \beta < \gamma < \delta$

$$K^{(p)} = \{ \alpha_0 < \alpha_1 < \cdots < \alpha_p \mid X_{\alpha_0} \cap \cdots \cap X_{\alpha_p} \neq \emptyset \}$$

Example.

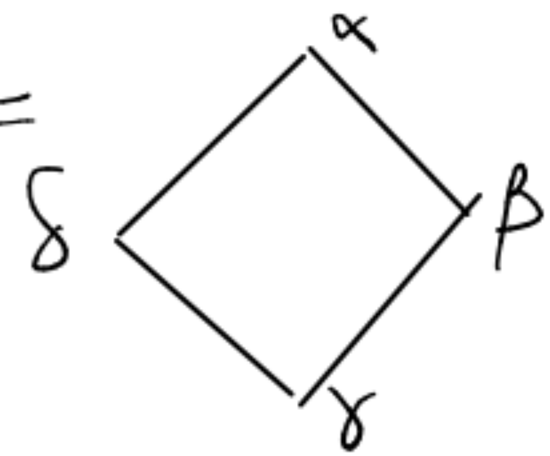
$X =$



$X_\alpha =$

$$K = \{ \alpha, \beta, \gamma, \delta, \alpha\beta, \beta\gamma, \gamma\delta, \delta\alpha, \}$$

$K =$



$X_{\alpha\beta} = \bullet\, 2$ (for example)

For each simplex $\sigma = \alpha_0 < \cdots < \alpha_p$

define $X_\sigma = X_{\alpha_0} \cap \cdots \cap X_{\alpha_p}$

$$C_p = \bigoplus_{\sigma \in K^{(p)}} C(X_\sigma)$$

This has as a basis simplices $u \in X_\sigma$, for each $\sigma \in K$

i.e. pairs (σ, u)

$$C_p = \bigoplus_{\sigma \in K^{(p)}} C(X_\sigma).$$

We define a complex of chain complexes

$$C(X) \leftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_p \leftarrow \cdots$$

If $\sigma = \alpha_0 < \cdots < \alpha_p$ and $0 \le i \le p$ put

$$\partial_i(\sigma) = \alpha_0 < \cdots < \hat{\alpha}_i < \cdots < \alpha_p$$

(a $p-1$ simplex)

We have inclusions $X_\sigma \hookrightarrow X_{\partial_i \sigma}$ giving a chain map $\partial_i : C(X_\sigma) \to C(X_{\partial_i \sigma})$

We define

$\partial : C_p \to C_{p-1}$ to be

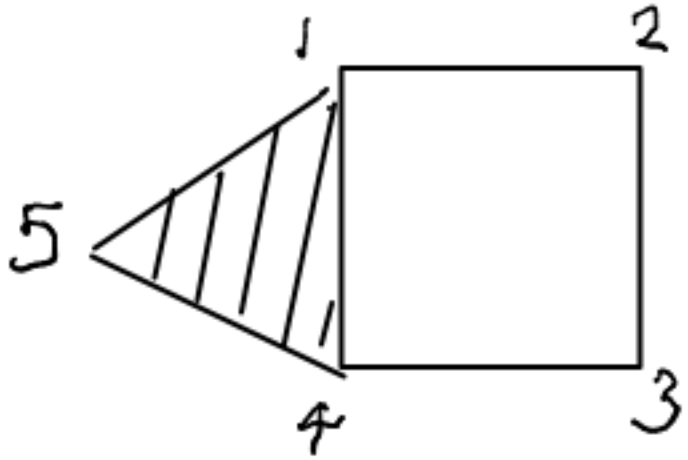
$$\partial = \sum_{i=0}^{p} (-1)^i \partial_i$$

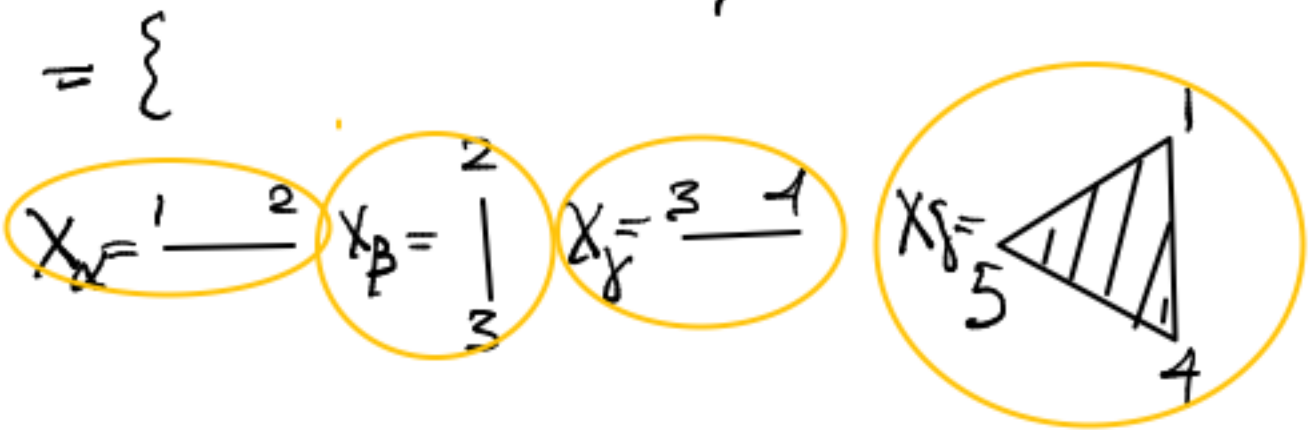
We also have inclusions $X_\alpha \xrightarrow{\iota_\alpha} X$ giving $\iota_\alpha : C(X_\alpha) \to C(X)$.

The sum of these is the map

$$C_0(X) = \bigoplus_{\alpha \in J} C(X_\alpha) \longrightarrow C(X).$$

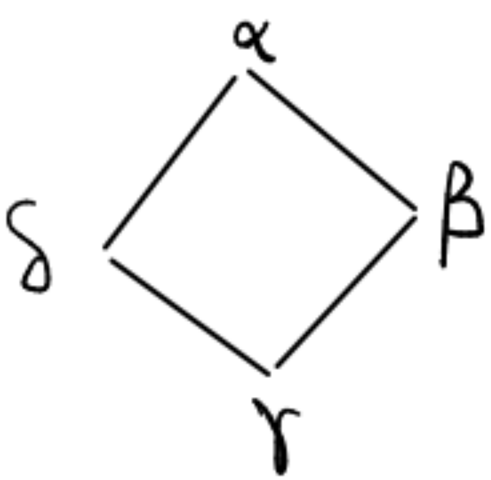
Example

$X =$ 

$= \{$ 

degree	0	1	2

$$C(X) = 0 \leftarrow R^5 \leftarrow R^6 \leftarrow R \leftarrow 0$$

$K =$ 

$$C_0 = C(X_\alpha) \oplus C(X_\beta) \oplus C(X_\gamma) \oplus C(X_\delta)$$

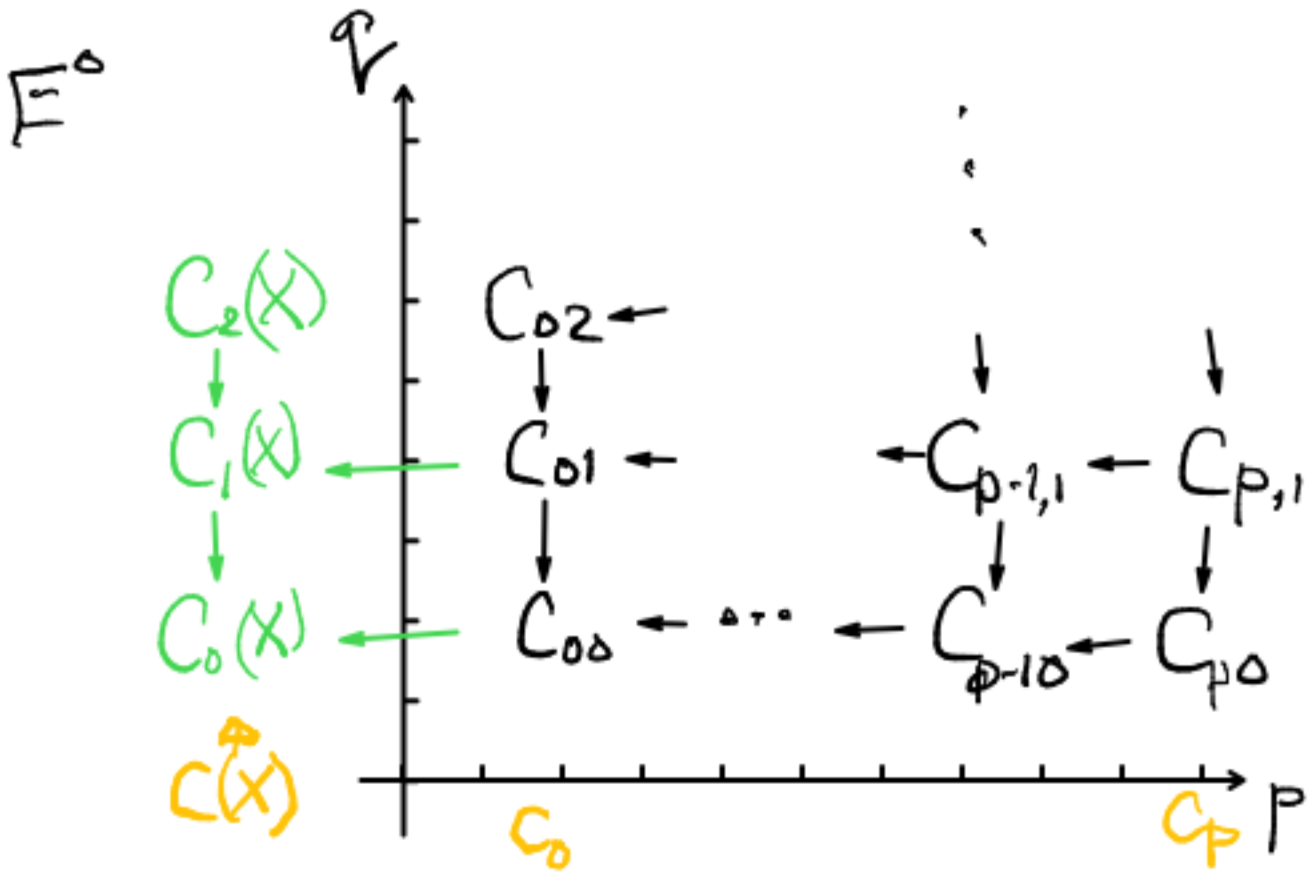
$$\begin{array}{l} \\ 1 \\ 0 \end{array} \quad = \begin{array}{c} \\ R \\ \downarrow \\ R^2 \end{array} \oplus \begin{array}{c} \\ R \\ \downarrow \\ R^2 \end{array} \oplus \begin{array}{c} \\ R \\ \downarrow \\ R^2 \end{array} \oplus \begin{array}{c} R \\ \downarrow \\ R^3 \\ \downarrow \\ R^3 \end{array}$$

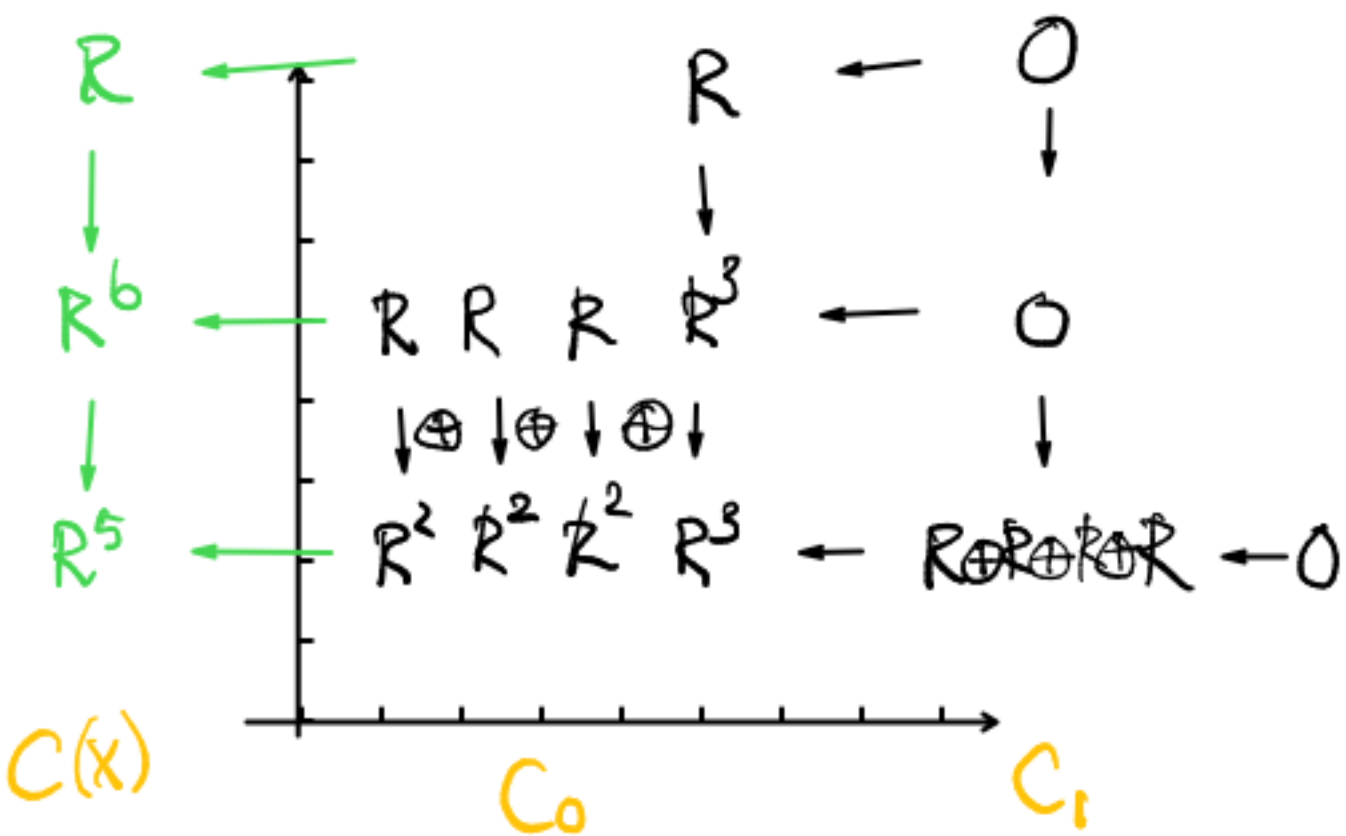
$$C_1 = C(X_{\alpha\beta}) \oplus C(X_{\beta\gamma}) \oplus C(X_{\gamma\delta}) \oplus C(X_{\delta\alpha})$$

$$0 \quad = R \oplus R \oplus R \oplus R$$

Example: the spectral sequence

We draw the sequence as a double complex





Lemma
The previously constructed complex (of chain complexes) is acyclic:

$$0 \leftarrow C(X) \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$$

Proof.
Recall $C_p = \bigoplus_{\sigma \in K^p} C(X_\sigma) = \begin{array}{c} C_{pq} \\ \downarrow \\ \vdots \\ \downarrow \\ C_{p1} \\ \downarrow \\ C_{p0} \end{array}$

1. We show the complex is acyclic in each degree q (as a chain complex of modules).

2 C_{pq} has a basis indexed by pairs (σ, u) where $\sigma \in K^{(p)}$, $u \in X_\sigma$ & a q-simplex

u arises each time it is a simplex in X_σ

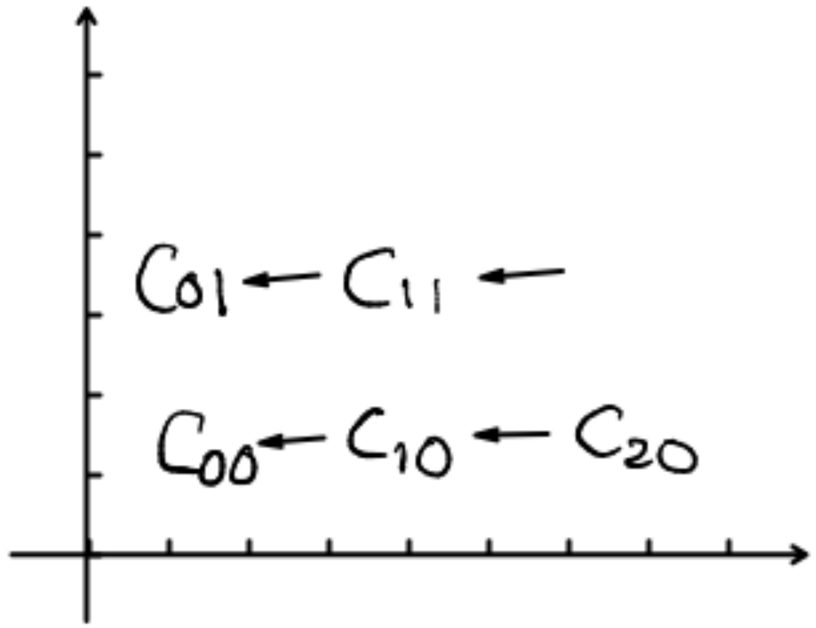
3. Fix a q-simplex u in X. The σ that arise in pairs (σ,u) are the faces (=subsets) of the single simplex whose vertices are $\{ \alpha \in J \mid u \text{ is in } X_\alpha \}$.

4. These basis elements (σ,u) with u fixed span a subcomplex isomorphic to the chain complex of the single simplex. It is contractible, so acyclic. The chain complex is the direct sum of these.

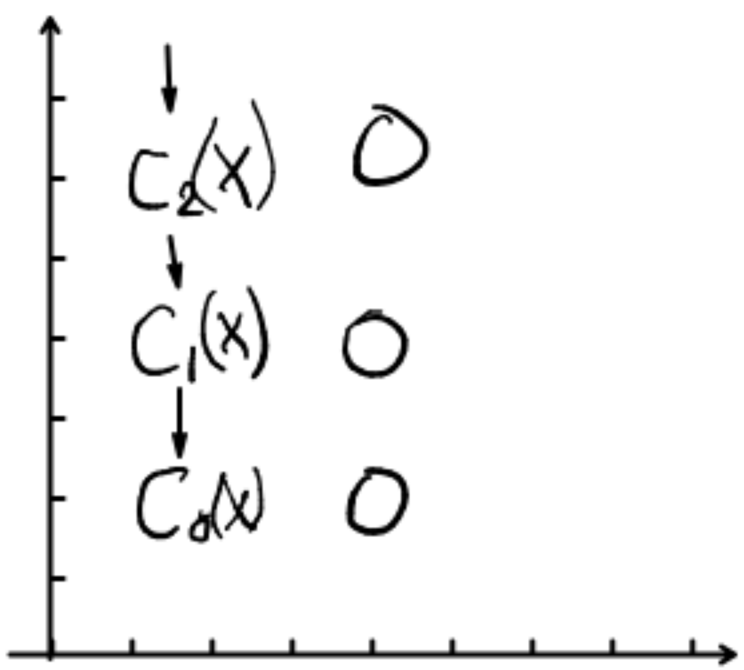
Filtering by rows

Proposition.

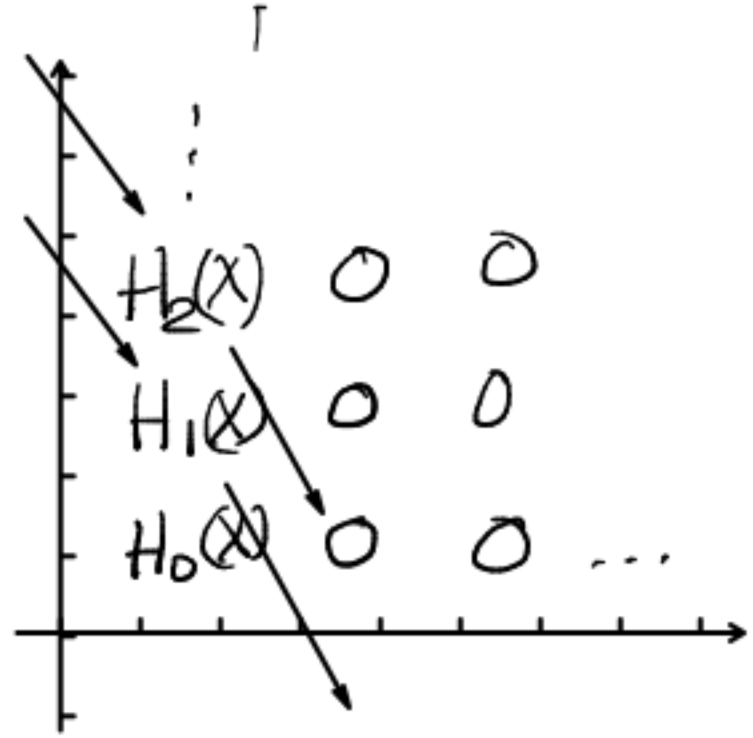
E^0



E^1



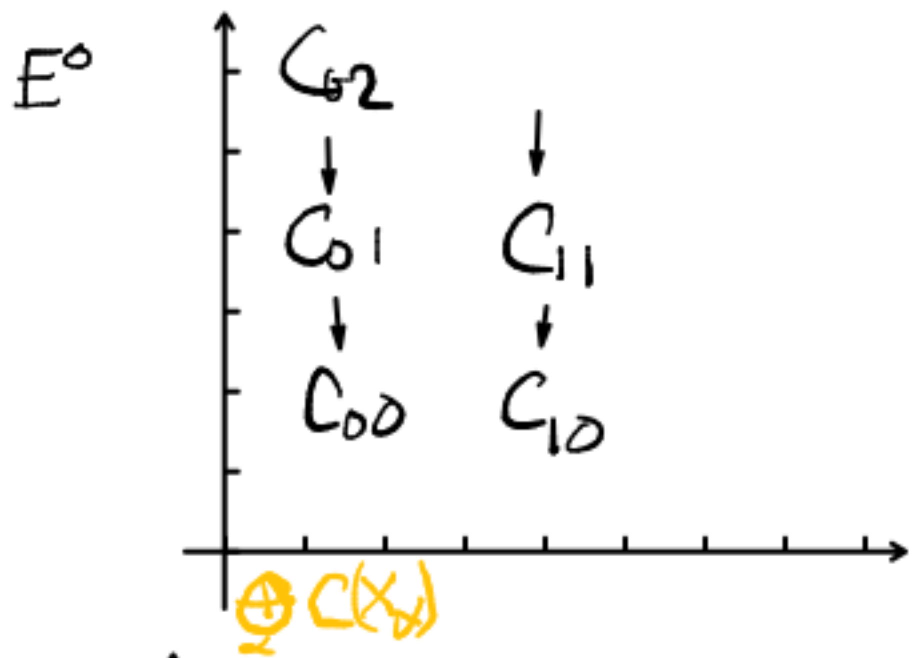
E^2

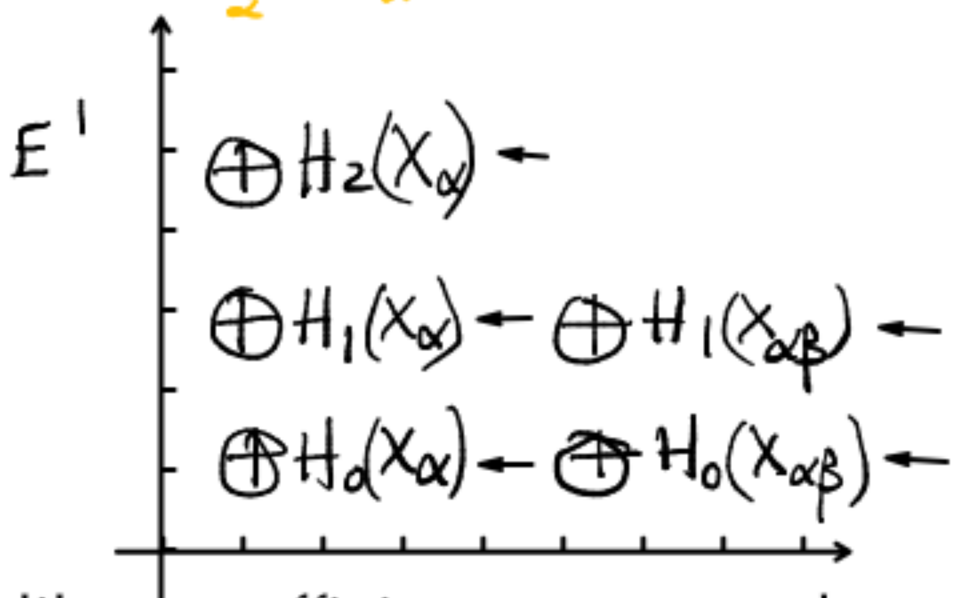


We conclude that $E^\infty = \mathrm{Gr}\ H_*(X)$

Filtering the double complex by columns

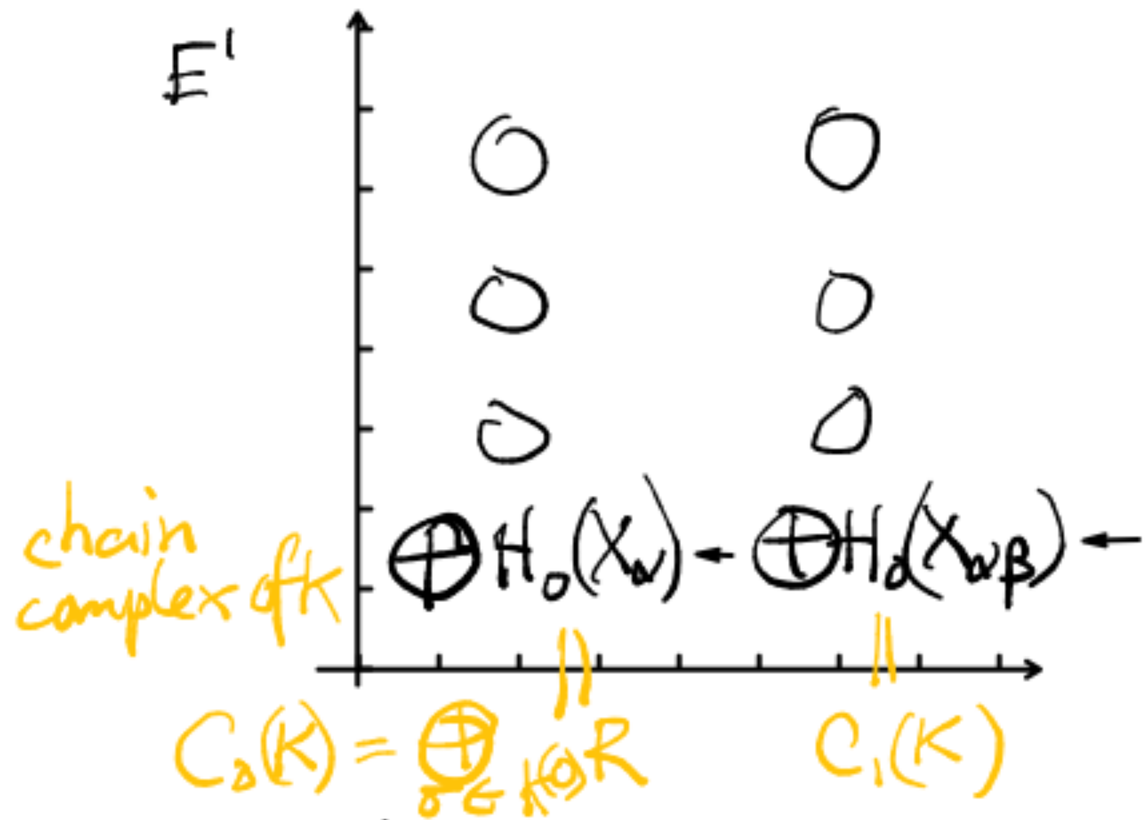
The spectral sequence looks like:

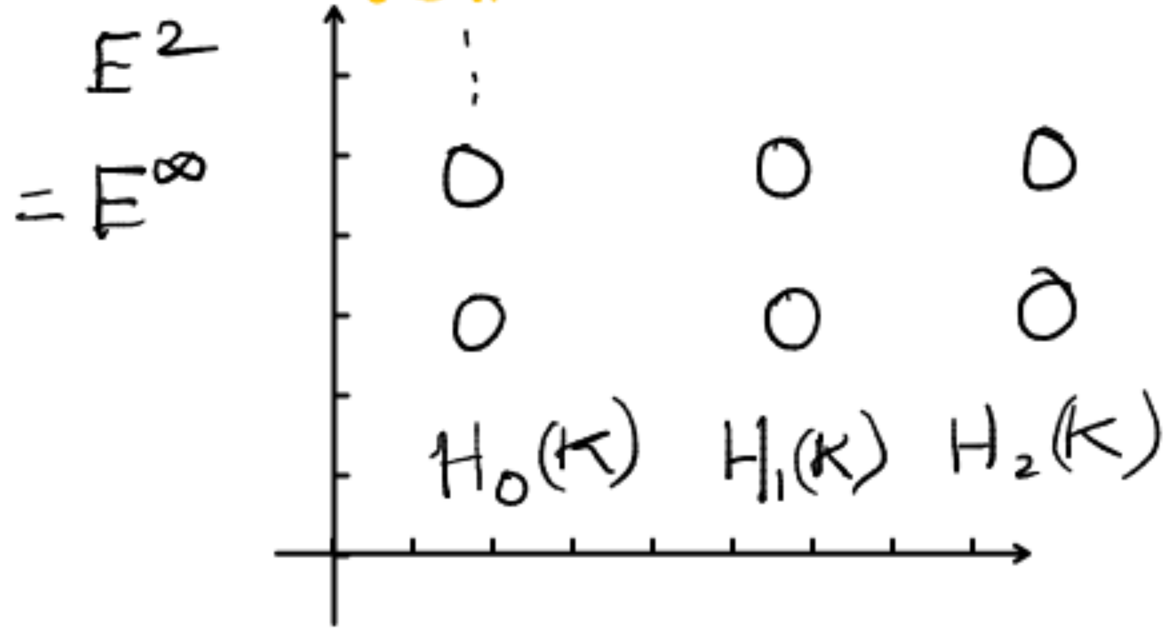




This looks like a coefficient system on the nerve of the covering.

Suppose that every non-empty intersection of the spaces in the covering is contractible. The E^1 page becomes:





Theorem. Suppose the simplicial complex X is the union of subcomplexes where every non-empty intersection is contractible. Then the homology of X is the same as the homology of the nerve of the covering (in a graded version).