Dimension theory and related things

1. Hilbert polynomials, Hilbert series, Poincare series

Definition. See the book of Atiyah – Macdonald.

A graded ring is a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A, such that

1. $A = \bigoplus_{n\geq 0} A_n$

2. $A_m A_n \subseteq A_{m+n}$

We see: each A_n is an A_0-module and A_0 is a subring.

Examples. 1. $A = k[x_1, \dots, x_r]$, A_n = set of homogeneous polynomials of degree n.

$$A = k[t^2, t^3] \subseteq k[t]$$

$A_0 = k$, $A_1 = 0$, $A_n = k \quad \forall n \geq 2$

3. If R is a ring, I is an ideal, put

$$A^* = \bigoplus_{n\geq 0} I^n, \quad A_n = I^n$$

$R = I^0 = A_0$.

4. The graded object associated to $I^j \subseteq I^{j-1} \subseteq \dots R$ we get

$$G(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$$

If $x_m \in I^m$, $x_n \in I^n$, write $\overline{x_m}$ = image in I^m/I^{m+1}, define $\overline{x_m} \cdot \overline{x_n} = \overline{x_m x_n}$

This is well-defined.

5 We could grade by a different monoid.

Pre-class Warm-up!!

Are you familiar with the formula for the dimension of the space of homogeneous polynomials in $k[x_1, \ldots, x_d]$ of degree n as

$$\binom{n+d-1}{d-1} \quad ?$$

A Yes

B No

Definition.
Let A be a graded ring
A graded A-module is an A-module M together with a family $(M_n)_{n \geq 0}$ of subgroups of M such that

1. $M = \bigoplus_{n \geq 0} M_n$

2. $A_m M_n \subseteq M_{m+n}$

An element $u \in M_m$ is called homogenous of degree m.
The subgroups M_m are the 'homogeneous components'

$A_+ = \bigoplus_{n \geq 1} A_n$ is an ideal of A.

More definitions:
Homogeneous elements, degree, homogeneous components, homomorphism of graded modules.

A_+

A homomorphism $\phi : L \to M$ is a homom. of graded modules if $\phi(L_m) \subseteq M_m$ $\forall m$.

Let A be commutative.

Proposition. TFAE for a graded ring A:

a. A is a Noetherian ring;

b. A_0 is Noetherian and A is finitely generated as an A_0-algebra.

Proof.

b => a is Hilbert's basis theorem. $A_0[u_1, \dots, u_s]$ is Noetherian, and A is an image of this.

a => b $A_0 = A/A_+$ is Noetherian.

The ideal A_+ is finitely generated, say by $x_1, \dots, x_s$. We may take these elements to be homogeneous. Why?

Let A' be the subring of A generated by $x_1, \dots, x_s$. and A_0 = A_0-subalgebra gen'd by $x_1, \dots, x_s$

We show that $A_n \subseteq A'$ for all $n \geq 0$.

Induction on n.

✓

For $n > 0$ let y be in A_n. Because y is in A_+ we can write y as a linear combination of the x_i, say

$$y = \sum_{i=1}^{s} a_i x_i$$

where $a_i \in A_{n-k_i}$

Let k_i be the degree of the homogeneous element x_i.

Each $k_i > 0$ so by induction each a_i is a polynomial in the x's with coefficients in A_0. The same is true of y, therefore y is in A'. Hence A_n is contained in A', so $A = A'$.

Hilbert functions

Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring, and A is generated (as an A_0-algebra) by elements $x_1, \ldots, x_s$ which we may choose to be homogeneous, of degrees k_1, ... , k_s

Let M be a finitely generated graded A-module, generated by homogeneous elements m_j, 1 ≤ j ≤ t. Each graded component M_n is now finitely generated as an A_0-module because M_n is generated as an A_0-module by elements $g_j(x) m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n - \deg m_j$.

Example $A = M = k[x_1, \ldots x_n]$
$P(M,t) = \frac{1}{(1-t)^n}$

Let λ: fin. gen A_0-modules $\to \mathbb{Z}$ be an 'additive function', meaning $\forall$ s.e.s. of A_0-modules $0 \to L \to M \to N \to 0$ we have $\lambda(M) = \lambda(L) + \lambda(N)$.

e.g. if $A_0 = k$ is a field $\lambda = \dim$ is possible
Or λ = composition length.

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n \geq 0} \lambda(M_n) t^n \quad \text{in } Z[[t]].$$

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Theorem (Hilbert, Serre)
Let A be a Noetherian graded ring, M a finitely generated graded A-module, λ a length function.
Then P(M,T) is a rational function in t of the form

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}, \quad f \in \mathbb{Z}[t]$$

Here $A = A_0[x_1, \dots, x_s]$, $\deg x_i = k_i$