Dimension theory and related things

1. Hilbert polynomials, Hilbert series, Poincare series

Definition. See the book of Atiyah – Macdonald.

A graded ring is a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A, such that

1. $A = \bigoplus_{n\geq 0} A_n$

2. $A_m A_n \subseteq A_{m+n}$

We see: each A_n is an A_0-module and A_0 is a subring.

Examples. 1. $A = k[x_1,\dots,x_r]$, A_n = set of homogeneous polynomials of degree n.

$A = k[t^2,t^3] \subseteq k[t]$

$A_0 = k$, $A_1 = 0$, $A_n = k$ $\forall n\geq 2$

3. If R is a ring, I is an ideal, put

$A^* = \bigoplus_{n\geq 0} I^n$, $A_n = I^n$

$R = I^0 = A_0$.

4. The graded object associated to $I^j \subseteq I^{j-1} \subseteq \dots R$ we get

$G(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$

If $x_m \in I^m$, $x_n \in I^n$, write

$\overline{x_m}$ = image in I^m/I^{m+1},

define $\overline{x_m}\cdot\overline{x_n} = \overline{x_m x_n}$

This is well-defined.

5 We could grade by a different monoid.

Pre-class Warm-up!!

Are you familiar with the formula for the dimension of the space of homogeneous polynomials in $k[x_1, \dots, x_d]$ of degree n as

$$\binom{n+d-1}{d-1} \quad ?$$

A Yes

B No

Definition.

Let A be a graded ring

A graded A-module is an A-module M together with a family $(M_n)_{n \geq 0}$ of subgroups of M such that

1. $M = \bigoplus_{n \geq 0} M_n$

2. $A_m M_n \subseteq M_{m+n}$

An element $u \in M_m$ is called homogenous of degree m.

The subgroups M_m are the 'homogeneous components'

$A_+ = \bigoplus_{n \geq 1} A_n$ is an ideal of A.

More definitions:

Homogeneous elements, degree, homogeneous components, homomorphism of graded modules.

A_+

A homomorphism $\phi : L \to M$ is a homom. of graded modules if $\phi(L_m) \subseteq M_m$ $\forall m$.

Let A be commutative.

Proposition. TFAE for a graded ring A:

a. A is a Noetherian ring;

b. A_0 is Noetherian and A is finitely generated as an A_0-algebra.

Proof.

b => a is Hilbert's basis theorem. $A_0[u_1, \ldots, u_s]$ is Noetherian, and A is an image of this.

a => b A_0 = A/A_+ is Noetherian.

The ideal A_+ is finitely generated, say by x_1, ... , x_s. We may take these elements to be homogeneous. Why?

Let A' be the subring of A generated by x_1, ... , x_s. and A_0 = A_0-subalgebra gen'd by $x_1, \ldots, x_s$

We show that A_n $\subseteq$ A' for all n ≥ 0.

Induction on n.

✓

For n > 0 let y be in A_n. Because y is in A_+ we can write y as a linear combination of the x_i, say

$$y = \sum_{i=1}^{s} a_i x_i$$

where $a_i \in A_{n-k_i}$

Let k_i be the degree of the homogeneous element x_i.

Each k_i > 0 so by induction each a_i is a polynomial in the x's with coefficients in A_0.

The same is true of y, therefore y is in A'.

Hence A_n is contained in A', so A = A'.

Hilbert functions

Let $A = \bigoplus_{n\geq 0} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring, and A is generated (as an A_0-algebra) by elements

$x_1, \dots, x_s$

which we may choose to be homogeneous, of degrees $k_1, \dots, k_s$

Let M be a finitely generated graded A-module, generated by homogeneous elements m_j, $1 \leq j \leq t$. Each graded component M_n is now finitely generated as an A_0-module because M_n is generated as an A_0-module by elements $g_j(x) m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n - \deg m_j$.

Example $A = M = k[x_1, \dots x_n]$

$P(M,t) = \frac{1}{(1-t)^n}$

Let λ: fin. gen A_0-modules $\to \mathbb{Z}$ be an 'additive function', meaning $\forall$ s.e.s. of A_0-modules $0 \to L \to M \to N \to 0$ we have $\lambda(M) = \lambda(L) + \lambda(N)$.

e.g. if $A_0 = k$ is a field $\lambda = \dim$ is possible. Or λ = composition length.

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n\geq 0} \lambda(M_n)\, t^n \quad \text{in } Z[[t]].$$

Pre-class Warm-up!

Is the following true/false, obvious/
not obvious ?

Let M be an R-module where R is a commutative ring, and let r be an element of R.

There is an exact sequence

$$0 \to K \to M \xrightarrow{r} M \to L \to 0$$

where the middle map is multiplication by r and both K and L are annihilated by r.

A false

B (probably) true and not obvious

$K \cong \ker(r)$ $L \cong M/rM$.

r kills K and L

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n \geq 0} \lambda(M_n)\, t^n \quad \text{in } \mathbb{Z}[[t]].$$

Theorem (Hilbert, Serre)
Let A be a Noetherian graded ring, M a finitely generated graded A-module, λ a length function.
Then P(M,T) is a rational function in t of the form

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}, \quad f \in \mathbb{Z}[t]$$

Here $A = A_0[x_1, \ldots, x_s]$, $\deg x_i = k_i$

Proof. Induction on s.
When $s = 0$, $A = A_0$ and M is only non-zero in finitely many degrees. $P(M,t)$ is a polynomial.
Now suppose $s > 0$ and result is true for smaller values. Consider the exact sequence

$$0 \to K \to M \xrightarrow{x_s} M \to L \to 0$$

x_s is supposed to be homogeneous so K and L are graded A-modules, finitely gen'd because A is Noetherian. They are killed by x_s. As $A_0[x_1, \ldots, x_{s-1}]$-modules they are finitely generated.
Thus $P(K,t)$ and $P(L,t)$ have the stated form

$$0 \to K \to M \xrightarrow{x_s} M \to L \to 0.$$

Also for each degree n.

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0.$$

Multiply by t^{n+k_s} and sum $\left(\sum_{n \geq 0}\right)$ we get

$$t^{k_s} P(K,t) - t^{k_s} P(M,t) + P(M,t) - P(L,t) = g(t)$$ for some polynomial $g(t)$

Rearrange:

$$(1 - t^{k_s}) P(M,t) = P(L,t) - t^{k_s} P(K,t) + g(t).$$

$$= \frac{\text{polynomial}}{\prod_{i=1}^{s-1} (1 - t^{k_i})}$$

Therefore

$$P(M,t) = \frac{\text{polynomial}}{\prod_{i=1}^{s} (1 - t^{k_i})}$$

□

Did we really need Hilbert and Serre to prove this?

Is it obvious why we need the polynomial g?

Corollary. A is a Noetherian graded ring generated as an A_0-algebra by homogeneous elements of degrees k_i. If each k_i = 1 then, for sufficiently large n, λ(M_n) is a polynomial in n (with rational coefficients) of degree d-1, where d is the order of the pole of P(M,t) at t = 1. $d = d(M)$.

Proof. Here $\lambda(M_n)$ = coefficient of t^n in function $\frac{f(t)}{(1-t)^s}$

$$= \frac{f_1(t)}{(1-t)^d}$$

where f, f_1 are polynomials

$$f = f_1 (1-t)^{s-d}$$

Write $f_1(t) = \sum_{k=0}^{N} a_k t^k$

Also $\frac{1}{(1-t)^d} = \sum_{u=0}^{\infty} \binom{d+u-1}{d-1} t^u$

$$\frac{f_1(t)}{(1-t)^d} = \left(\sum_{k=0}^{N} a_k t^k\right)\left(\sum \binom{d+u-1}{d-1} t^u\right)$$

has, for $n \geq N$, coeff of t^n

$$\sum_{k=0}^{N} a_k \binom{d+(n-k)-1}{d-1}.$$

This is a polynomial in n with leading term $\left(\sum_{k=0}^{N} a_k\right)\frac{n^{d-1}}{(d-1)!} \neq 0$

□

Discussion: Is anything about that at all remarkable?

Definition. The polynomial just described is the Hilbert function (or polynomial) of M.

Write $d(M)$ for the order of the pole of $P(M,t)$ at 1.

Corollary. If x in A is not a zero divisor on M then $d(M/xM) = d(M) - 1$.

Not a zero divisor means $xm = 0$ implies $m = 0$.

Proof.

Examples.

2. A = k[t^2, t^3]

What is the Poincare series of A (with respect to the k-dimension of terms)?

A. $\frac{1+t^2}{1-t}$

B. $\frac{1-t+t^2}{1-t} = \frac{1+t^3}{1-t^2}$

$\frac{1+t+t^2}{1-t} = \frac{1-t^3}{(1-t)^2}$

D None of the above.

The graded ring associated to an ideal

Proposition.
Let J be an ideal of a Noetherian ring A.
Then the graded ring

$$G(A) = \bigoplus_{n=0}^{\infty} J^n / J^{n+1}$$

is Noetherian, generated by elements of degree 1.
If M is a finitely generated A-module then

$$G(M) = \bigoplus_{n=0}^{\infty} J^n M / J^{n+1} M$$

is a finitely generated G(A)-module.

Proof

The same is true for G(M) if it is defined by a filtration that eventually is multiplication by J.

We look for a situation where J is an ideal of A for which there is a suitable additive function on A/J-modules.

If J is primary for some maximal ideal of A then A/J is Artinian.

Proposition.
Let J be an ideal of A so that A/J is Artinian, let M be a finitely generated A-module. Then
a. M/J^nM is of finite length for each $n \geq 0$.
b. For all sufficiently large n this length is a polynomial $g(n)$ of degree $\leq s$ in n where s is the least number of generators of J.

Proof.