Dimension theory and related things

1. Hilbert polynomials, Hilbert series, Poincare series

Definition. See the book of Atiyah – Macdonald.

A graded ring is a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A, such that

1. $A = \bigoplus_{n\geq 0} A_n$

2. $A_m A_n \subseteq A_{m+n}$

We see: each A_n is an A_0-module and A_0 is a subring.

Examples. 1. $A = k[x_1,\dots,x_r]$, A_n = set of homogeneous polynomials of degree n.

$A = k[t^2,t^3] \subseteq k[t]$

$A_0 = k$, $A_1 = 0$, $A_n = k$ $\forall n\geq 2$

3. If R is a ring, I is an ideal, put

$A^* = \bigoplus_{n\geq 0} I^n$, $A_n = I^n$

$R = I^0 = A_0$.

4. The graded object associated to $I^j \subseteq I^{j-1} \subseteq \cdots R$ we get

$G(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$

If $x_m \in I^m$, $x_n \in I^n$, write

$\overline{x_m}$ = image in I^m/I^{m+1},

define $\overline{x_m}\cdot\overline{x_n} = \overline{x_m x_n}$

This is well-defined.

5 We could grade by a different monoid.

Pre-class Warm-up!!

Are you familiar with the formula for the dimension of the space of homogeneous polynomials in k[x_1, ... , x_d] of degree n as

$$\binom{n+d-1}{d-1} \quad ?$$

A Yes

B No

Definition.

Let A be a graded ring

A graded A-module is an A-module M together with a family (M_n)$_{n \geq 0}$ of subgroups of M such that

1. $M = \bigoplus_{n \geq 0} M_n$

2. $A_m M_n \subseteq M_{m+n}$

An element $u \in M_m$ is called homogenous of degree m.

The subgroups M_m are the 'homogeneous components'

$A_+ = \bigoplus_{n \geq 1} A_n$ is an ideal of A.

More definitions:

Homogeneous elements, degree, homogeneous components, homomorphism of graded modules.

A_+

A homomorphism $\phi : L \to M$ is a homom. of graded modules if $\phi(L_m) \subseteq M_m$ $\forall m$.

Let A be commutative.

Proposition. TFAE for a graded ring A:

a. A is a Noetherian ring;

b. A_0 is Noetherian and A is finitely generated as an A_0-algebra.

Proof.

b => a is Hilbert's basis theorem. $A_0[u_1, \ldots, u_s]$ is Noetherian, and A is an image of this.

a => b A_0 = A/A_+ is Noetherian.

The ideal A_+ is finitely generated, say by x_1, … , x_s. We may take these elements to be homogeneous. Why?

Let A′ be the subring of A generated by x_1, … , x_s. and A_0 = A_0-subalgebra gen'd by $x_1, \ldots, x_s$

We show that A_n $\subseteq$ A′ for all n ≥ 0.

Induction on n.

✓

For n > 0 let y be in A_n. Because y is in A_+ we can write y as a linear combination of the x_i, say

$$y = \sum_{i=1}^{s} a_i x_i$$

where $a_i \in A_{n-k_i}$

Let k_i be the degree of the homogeneous element x_i.

Each k_i > 0 so by induction each a_i is a polynomial in the x's with coefficients in A_0.

The same is true of y, therefore y is in A′.

Hence A_n is contained in A′, so A = A′.

Hilbert functions

Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring, and A is generated (as an A_0-algebra) by elements

$x_1, \ldots, x_s$

which we may choose to be homogeneous, of degrees k_1, ... , k_s

Let M be a finitely generated graded A-module, generated by homogeneous elements m_j, 1 ≤ j ≤ t. Each graded component M_n is now finitely generated as an A_0-module because M_n is generated as an A_0-module by elements $g_j(x) m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n - \deg m_j$.

Example $A = M = k[x_1, \ldots x_n]$

$P(M,t) = \frac{1}{(1-t)^n}$

Let λ: fin. gen A_0-modules $\to \mathbb{Z}$ be an 'additive function', meaning $\forall$ s.e.s. of A_0-modules $0 \to L \to M \to N \to 0$ we have $\lambda(M) = \lambda(L) + \lambda(N)$.

e.g. if $A_0 = k$ is a field $\lambda = \dim$ is possible

Or λ = composition length.

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n \geq 0} \lambda(M_n) t^n \quad \text{in } Z[[t]].$$

Pre-class Warm-up!

Is the following true/false, obvious/
not obvious ?

Let M be an R-module where R is a commutative ring, and let r be an element of R.

There is an exact sequence

$$0 \to K \to M \xrightarrow{r} M \to L \to 0$$

where the middle map is multiplication by r and both K and L are annihilated by r.

A false

B (probably) true and not obvious

$K \cong \ker(r) \quad L \cong M/rM$.

r kills K and L

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n \geq 0} \lambda(M_n)\, t^n \quad \text{in } Z[[t]].$$

Theorem (Hilbert, Serre)
Let A be a Noetherian graded ring, M a finitely generated graded A-module, λ a length function.
Then P(M,T) is a rational function in t of the form

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}, \quad f \in \mathbb{Z}[t]$$

Here $A = A_0[x_1, \ldots, x_s]$, $\deg x_i = k_i$

Proof. Induction on s.
When $s = 0$, $A = A_0$ and M is only non-zero in finitely many degrees. $P(M,t)$ is a polynomial.
Now suppose $s > 0$ and result is true for smaller values. Consider the exact sequence

$$0 \to K \to M \xrightarrow{x_s} M \to L \to 0$$

x_s is supposed to be homogeneous so K and L are graded A-modules, finitely gen'd because A is Noetherian. They are killed by x_s. As $A_0[x_1, \ldots, x_{s-1}]$-modules they are finitely generated.
Thus $P(K,t)$ and $P(L,t)$ have the stated form

$$0 \to K \to M \xrightarrow{x_s} M \to L \to 0.$$

Also for each degree n.

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0.$$

Multiply by t^{n+k_s} and sum $\left(\sum_{n\geq 0}\right)$ we get

$$t^{k_s}P(K,t) - t^{k_s}P(M,t) + P(M,t) - P(L,t) = g(t)$$ for some polynomial $g(t)$

Rearrange:

$$(1-t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t).$$

$$= \frac{\text{polynomial}}{\prod_{i=1}^{s-1}(1-t^{k_i})}$$

Therefore

$$P(M,t) = \frac{\text{polynomial}}{\prod_{i=1}^{s}(1-t^{k_i})}$$

$\square$

Did we really need Hilbert and Serre to prove this?

Is it obvious why we need the polynomial g?

Corollary. A is a Noetherian graded ring generated as an A_0-algebra by homogeneous elements of degrees k_i. If each k_i = 1 then, for sufficiently large n, $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree d-1, where d is the order of the pole of P(M,t) at t = 1. $d = d(M)$.

Proof. Here $\lambda(M_n)$ = coefficient of t^n in function $\frac{f(t)}{(1-t)^s}$

$$= \frac{f_1(t)}{(1-t)^d}$$

where f, f_1 are polynomials

$$f = f_1 (1-t)^{s-d}$$

Write $f_1(t) = \sum_{k=0}^{N} a_k t^k$

Also $\frac{1}{(1-t)^d} = \sum_{u=0}^{\infty} \binom{d+u-1}{d-1} t^u$

$$\frac{f_1(t)}{(1-t)^d} = \left(\sum_{k=0}^{N} a_k t^k\right)\left(\sum \binom{d+u-1}{d-1} t^u\right)$$

has, for $n \geq N$, coeff of t^n

$$\sum_{k=0}^{N} a_k \binom{d+(n-k)-1}{d-1}.$$

This is a polynomial in n with leading term $\left(\sum_{k=0}^{N} a_k\right)\frac{n^{d-1}}{(d-1)!} \neq 0$

$\square$

Discussion: Is anything about that at all remarkable?

Definition. The polynomial just described is the Hilbert function (or polynomial) of M.

Pre-class Warm-up!!

Let d be the order of the pole of

$$\frac{1+t+t^2}{(1-t)^d} = \sum_{n\geq 0} a_n t^n$$

at t = 1. Which of the following correctly describes the degree of polynomial growth of the coefficients a_n as n increases?

A d - 1 ✓

B d

C d + 1

D None of the above.

$$\frac{1}{1-t} = 1+t+t^2+\cdots$$

$a_n = 1$ always

polynomial of degree 0.

$$\frac{1}{(1-t)^2} = 1+2t+3t^2+\cdots$$

$$= \sum_{n\geq 0} \binom{n+1}{1} t^n.$$

$a_n = n+1$ polynomial of degree 1.

$$\frac{1}{1-t^2} = 1+t^2+t^4+t^6+\cdots$$

a_n sequence is 1, 0, 1, 0, 1, 0, ...

not polynomial. This is described by two polynomials $f_0(n) = 1$, $f_1(n) = 0$

If $n \equiv i \pmod{2}$ then $a_n = f_i(n)$

These a_n are (almost) Polynomial On Residue Classes

= (almost) PORC

Write d(M) for the order of the pole of P(M,t) at 1. M is a f g graded A-module

A is always Noetherian. A is graded now

Corollary. If a homogeneous element x in A is not a zero divisor on M then d(M/xM) = d(M) - 1.

Not a zero divisor means xm = 0 implies m = 0. This happens e.g. if A = M is a domain and $x \neq 0$.

Proof. We had an exact sequence

$$0 \to K \to M \xrightarrow{x} M \to L \to 0,$$

x is not a zero divisor $\iff K = 0$,

$L = M/xM$, also.

s.e.s. $0 \to M_n \to M_{n+r} \to L_{n+r} \to 0$

r = degree of x. Multiply by t^{n+1}, sum

$$t^r P(M,t) - P(M,t) + P(L,t) = g(t)$$

$$P(L,t) = (1-t^r)P(M,t) + g(t)$$

$$d(L) = d(M) - 1$$

$\square$

There is a more subtle version of this in the context of local rings where we remove the assumption that x is homogeneous.

Examples.

$= A$ graded as usual.

$$P(A,t) = \frac{1}{(1-t)^s}.$$

$$1 + \frac{t^2}{1-t} = \frac{1-t+t^2}{1-t}$$

2. A = k[t^2, t^3] $= k \cdot 1 \oplus 0 \oplus k t^2 \oplus k \cdot t^3 \oplus \ldots$

What is the Poincare series of A $P(A,t)$ (with respect to the k-dimension of terms)?

A. $\frac{1+t^2}{1-t}$

B. $\frac{1-t+t^2}{1-t} = \frac{1+t^3}{1-t^2}$

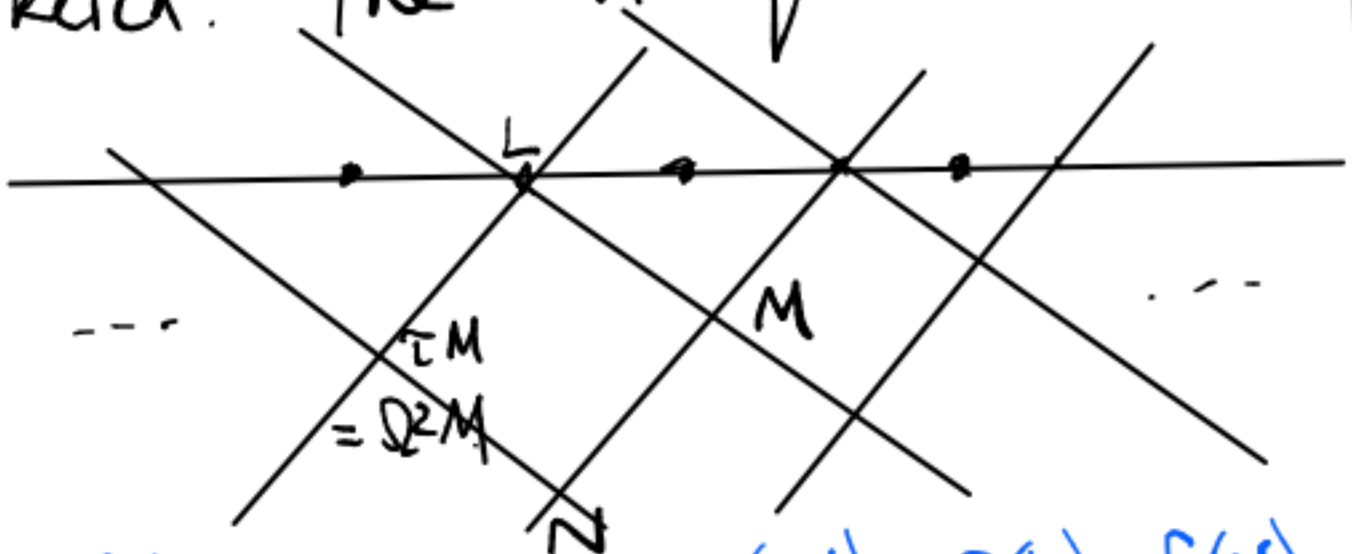
$\frac{1+t+t^2}{1-t} = \frac{1-t^3}{(1-t)^2}$

D None of the above.

The Auslander-Reiten quiver of a finite group. Math. Z. 1982

Theorem k a field, G a finite group. The tree class of any cmpt of the AR quiver of kG is an (extended) Dynkin diagram or one of 5 infinite trees.

Sketch. The AR quiver is



additive means $f(M) + f(\tau M) = f(L) + f(N)$

$\Omega^i M$ = kernel K_{i-1} in a minimal projective resolution of M.

$\dim \Omega^{2i} M$ is an almost PORC function of i.

This comes about because $H^*(G,k)$ is a finitely generated graded-commutative ring (Evens–Venkov)

$H^*(G,M)$ is a finitely generated graded $H^*(G,k)$-module.

$\dim(\Omega^i M)$ is determined by $H^*(G,M)$.

$M \mapsto f(M)$ = leading coefficient of the PORC polynomial. is a periodic additive function on the AR-quiver. Happel–Preiser–Ringel showed this $\Rightarrow$ the tree class is so.

Pre-class Warm-up!!

Noetherian

Let J be an ideal of a commutative ring R.
Consider the two statements:

1. If R/J is Artinian then J is primary for some maximal ideal of R.

2. If J is primary for some maximal ideal of R then R/J is Artinian.

Which are true?

A 1. is true. × $R = k \times k$, $J = (0)$

B 2. is true. ✓

C Both 1. and 2. are true.

D Neither are true.

Artinian : DCC on submodules
⇒ f.g. modules have composition series

Fin. dim algebras over a field are Artinian

primary = one associated prime only

Fact: minimal primes containing J are associated primes.
Conclude (from J-primary, $\mathfrak{m}$ maxl)
$\mathfrak{m}$ is the unique minimal prime $= \sqrt{J}$, so $\mathfrak{m}^n \subseteq J$ for some n ⇒ R/J is Artinian (it has a compn series).

The graded ring associated to an ideal

Proposition. commutative

Let J be an ideal of a Noetherian ring A. Then the graded ring

$$G(A) = \bigoplus_{n=0}^{\infty} J^n / J^{n+1}$$

is Noetherian, generated by elements of degree 1. as an A/J-algebra.

If M is a finitely generated A-module then

$$G(M) = \bigoplus_{n=0}^{\infty} J^n M / J^{n+1} M$$

is a finitely generated G(A)-module.

Proof A is Noetherian so J is a finitely generated ideal $J = (x_1, \dots, x_s)$

Let $\bar{x}_i$ = image of x_i in J/J^2
$= x_i + J^2$.

Then $G(A) = A/J[\bar{x}_1, \dots, \bar{x}_s]$.

Why?

Also A/J is Noetherian, so G(A) is Noetherian by Hilbert's basis theorem.

If M is a f.g. A-module then M/JM is a f.g. A/J-module.

and $J^n M / J^{n+1} M = (J^n / J^{n+1}) \cdot M/JM$

so M/JM generates G(M) as a G(A)-module. $\square$

Why: we show $J^n/J^{n+1} \subseteq$ right side $\forall n$. If $u \in J^n$ then $u + J^{n+1}$ is an A/J combination of products of the $\bar{x}_1, \dots, \bar{x}_s$.

The same is true for G(M) if it is defined by a filtration that eventually is multiplication by J and has J M_i contained in M_{i+1}. Such is called J-stable.

We look for a situation where J is an ideal of A for which there is a suitable additive function on A/J-modules.

If J is primary for some maximal ideal of A then A/J is Artinian. Take composition length over A/J as our 'additive function'

Proposition.
Let J be an ideal of A so that A/J is Artinian, let M be a finitely generated A-module. Then
a. M/J^nM is of finite length for each $n \geq 0$.
b. For all sufficiently large n this length is a polynomial g(n) of degree $\leq$ s in n where s is the least number of generators of J.

Proof. We have seen: each $J^n M / J^{n+1} M$ is finitely generated as an A/J-module so has finite length, given by a Hilbert polynomial for $n \gg 0$.

The length of $M/J^n M$ is
$$\sum_{t=0}^{n-1} \text{length}\left(J^t M / J^{t+1} M\right)$$
$$= \sum_{t=0}^{n-1} H(t) + \text{constant}, \quad n \gg 0$$
where H is polynomial of degree $\leq s-1$

Thus length $(M/J^n M)$ is polynomial of degree $\leq s$ for $n \gg 0$. $\square$

The Proposition works for a filtration of M that is J-stable. Part c. says the degree and leading coefficient of g(n) do not depend on the filtration chosen.
This is because
$$M_i \supseteq J^i M \supseteq M_{N+i} = J^i M_N$$
where $M_i = J M_{i-1}$ if $i \geq N$.

Definition.
Aliyah-Macdonald and Matsumura write the polynomial

$$g(n) = \chi_J^M(n) = \text{length}\left(M/J^n M\right) \quad n \gg 0$$

Matsumura calls it the Samuel function. When $M = A$, Aliyah-Macdonald call it the characteristic polynomial of the ideal J.

Corollary.
For large n, length (A J^n) is a polynomial of degree $\leq s$, where s = least number of

Proposition.
If J is $\mathfrak{m}$-primary where $\mathfrak{m}$ is a maximal ideal then

$$\deg \chi_J(n) = \deg \chi_{\mathfrak{m}}(n)$$

Definition. d(A) is

Let A be a (Noetherian) ring. We already

We have seen:
Dim A = 0 <=> A is Artinian.
Dimension is preserved under integral

Goal: Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$

Let $\partial(A)$ = least number of generators of an $\mathfrak{m}$-primary ideal of A.

We indicate a proof that

$$\partial(A) = d(A) = \dim(A)$$

Goal: Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$

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$$\partial(A) = d(A) = \dim(A)$$

showing $\partial(A) \geq d(A) \geq \dim A \geq \partial(A)$

Proposition.

Proposition 11.8 of Atiyah and Macdonald.
Notation as before.
Let M be a finitely-generated A-module, x in A a non-zero-divisor on M and M′ = M/xM.
Then

$$\deg \chi_J^{M'} \leq \deg \chi_J^{M} - 1$$

We did this before in a graded situation

Proof. Let N = xM, which is isomorphic to M.
We have a s.e.s.

0 -> N/(N ∩ J^nM) -> M/J^nM -> M′/J^nM′ -> 0

Writing g(n) = length of N/(N ∩ J^nM) we have

$g(n) - \chi_J^{M}(n) + \chi_J^{M'}(n) = 0$ For n >> 0

Artin-Rees implies that (N ∩ J^N) is a stable J-filtration of N, so g(n) and $\chi_J^{M}(n)$ have the same leading term. Hence the result.

Corollary 11.9 of A and M.
If A is a Noetherian local ring and x is a non-zero-divisor in A, then

$$d(A/(x)) \leq d(A) - 1.$$

Proposition 11.10 $d(A) \geq \dim A$

Can you remember what $d(A)$ is? What did the last result say?

Proof. Induction on $d = d(A)$.

If $d = 0$ then Length $(A/\mathfrak{m}^n)$ is constant for all large n, so $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n, hence $\mathfrak{m}^n = 0$ by Nakayama's lemma. Thus A is an Artinian ring and $\dim A = 0$

Suppose $d > 0$ and the result for smaller values. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$

be any chain of prime ideals in A.
Let $x \in \mathfrak{p}_1$, $x \notin \mathfrak{p}_0$, $A' = A/\mathfrak{p}_0$.

Let x' be the image of x in A'.
Then $x' \neq 0$, and A' is an integral domain, so by 11.9

$d(A' / (x')) \leq d(A') - 1$

Also, if $\mathfrak{m}'$ Is the maximal ideal of A', $A'/\mathfrak{m}'^n$ is a homomorphism image of $A/\mathfrak{m}^n$ hence Length$(A/\mathfrak{m}^n) \geq$ Length $(A'/\mathfrak{m}'^n)$ and therefore $d(A) \geq d(A')$. Thus

$d(A' / (x')) \leq d(A) - 1 = d - 1$.
By induction, the length of any chain of prime ideals in $A' / (x')$ is $\leq d - 1$.
The images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$
in $A' / (x')$ form a chain of length $r - 1$, hence $r - 1 \leq d - 1$ and consequently $r \leq d$. Hence $\dim A \leq d$.

The final step $\dim A \geq \partial(A)$ in proving $\partial(A) = d(A) = \dim A$ is a little technical and we are going to miss it out. The next corollaries do not depend on it.