

Dimension theory and related things

1. Hilbert polynomials, Hilbert series, Poincare series

Definition. *See the book of Atiyah - Macdonald.*

A graded ring is a ring A together with a family $(A_n)_{n \geq 0}$ of subgroups of the additive group of A , such that

$$1. A = \bigoplus_{n \geq 0} A_n$$

$$2. A_m A_n \subseteq A_{m+n}$$

We see: each A_n is an A_0 -module.

Examples. 1. and A_0 is a subring.

$A = k[x_1, \dots, x_r]$, $A_n =$ set of homogeneous polynomials of degree n .

$$A = k[t^2, t^3] \subseteq k[t]$$

$$A_0 = k, A_1 = 0, A_n = k \quad \forall n \geq 2$$

3. If R is a ring, I is an ideal, put

$$A^* = \bigoplus_{n \geq 0} I^n, \quad A_n = I^n$$
$$R = I^0 = A_0.$$

4. The graded object associated to $I^j \subseteq I^{j-1} \subseteq \dots \subseteq R$ we get

$$G(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}$$

If $x_m \in I^m$, $x_n \in I^n$, write $\bar{x}_m =$ image in I^m / I^{m+1} ,

define $\bar{x}_m \cdot \bar{x}_n = \overline{x_m x_n}$
This is well-defined.

5 We could grade by a different monoid.

Pre-class Warm-up!!

Are you familiar with the formula for the dimension of the space of homogeneous polynomials in $k[x_1, \dots, x_d]$ of degree n as

$$\binom{n+d-1}{d-1} ?$$

A Yes

B No

Definition.

Let A be a graded ring

A graded A -module is an A -module M together with a family $(M_n)_{n \geq 0}$ of subgroups of M such that

$$1. \quad M = \bigoplus_{n \geq 0} M_n$$

$$2. \quad A_m M_n \subseteq M_{m+n}$$

An element $u \in M_m$ is called homogeneous of degree m .

The subgroups M_m are the 'homogeneous components'

$A_+ = \bigoplus_{n \geq 1} A_n$ is an ideal of A .

More definitions:

Homogeneous elements, degree, homogeneous components, homomorphism of graded modules.

A_+

A homomorphism

$$\phi: L \rightarrow M$$

is a homom. of graded modules if $\phi(L_m) \subseteq M_m$.

$\forall m$.

Let A be commutative.

Proposition. TFAE for a graded ring A :

- A is a Noetherian ring;
- A_0 is Noetherian and A is finitely generated as an A_0 -algebra.

Proof.

$b \Rightarrow a$ is Hilbert's basis theorem. $A_0[x_1, \dots, x_s]$ is Noetherian, and A is an image of this.

$a \Rightarrow b$ $A_0 = A/A_+$ is Noetherian.

The ideal A_+ is finitely generated, say by x_1, \dots, x_s . We may take these elements to be homogeneous. Why?

Let A' be the subring of A generated by x_1, \dots, x_s and $A_0 = A_0$ -subalgebra gen'd by x_1, \dots, x_s .

We show that $A_n \subseteq A'$ for all $n \geq 0$.

Induction on n .



For $n > 0$ let y be in A_n . Because y is in A_+ we can write y as a linear combination of the x_i , say

$$y = \sum_{i=1}^s a_i x_i$$

where $a_i \in A_{n-k_i}$

Let k_i be the degree of the homogeneous element x_i .

Each $k_i > 0$ so by induction each a_i is a polynomial in the x 's with coefficients in A_0 .

The same is true of y , therefore y is in A' .

Hence A_n is contained in A' , so $A = A'$.

Hilbert functions

Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring, and A is generated (as an A_0 -algebra) by elements x_1, \dots, x_s

which we may choose to be homogeneous, of degrees k_1, \dots, k_s

Let M be a finitely generated graded A -module, generated by homogeneous elements $m_j, 1 \leq j \leq t$. Each graded component M_n is now finitely generated as an A_0 -module

because M_n is generated as an A_0 -module by elements $g_j(x) m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n - \deg m_j$.

Example $A = M_1 = k[x_1, \dots, x_n]$
 $P(M, t) = \frac{1}{(1-t)^n}$

Let $\lambda: \text{fin. gen } A_0\text{-modules} \rightarrow \mathbb{Z}$ be an 'additive functor', meaning \forall s.e.s. of A_0 -modules

$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ we have $\lambda(M) = \lambda(L) + \lambda(N)$.

e.g. if $A_0 = k$ is a field $\lambda = \dim$ is possible
Or $\lambda = \text{composition length}$.

Definition. The Poincaré series of M (with respect to λ) is

$$P(M, t) = \sum_{n \geq 0} \lambda(M_n) t^n \quad \text{in } \mathbb{Z}[[t]].$$

Pre-class Warm-up!

Is the following true/false, obvious/not obvious?

Let M be an R -module where R is a commutative ring, and let r be an element of R .

There is an exact sequence

$$0 \rightarrow K \rightarrow M \xrightarrow{r} M \rightarrow L \rightarrow 0$$

where the middle map is multiplication by r and both K and L are annihilated by r .

A false

B (probably) true and not obvious

$$K \cong \ker(r) \quad L \cong M/rM$$

r kills K and L

Definition. The Poincaré series of M (with respect to λ) is

$$P(M, t) = \sum_{n \geq 0} \lambda(M_n) t^n \quad \text{in } \mathbb{Z}[[t]].$$

Theorem (Hilbert, Serre)

Let A be a Noetherian graded ring, M a finitely generated graded A -module, λ a length function.

Then $P(M, t)$ is a rational function in t of the form

$$P(M, t) = \frac{f(t)}{\prod_{i=1}^s (1 - t^{k_i})}, \quad f \in \mathbb{Z}[t]$$

Here $A = A_0[x_1, \dots, x_s]$, $\deg x_i = k_i$

Proof. Induction on s .

When $s = 0$, $A = A_0$ and M is only non-zero in finitely many

degrees. $P(M, t)$ is a polynomial.

Now suppose $s > 0$ and result is true for smaller values. Consider the exact sequence

$$0 \rightarrow K \rightarrow M \xrightarrow{x_s} M \rightarrow L \rightarrow 0$$

x_s is supposed to be homogeneous so K and L are graded A -modules, finitely gen'd because A is Noetherian. They are killed by x_s . As $A_0[x_1, \dots, x_{s-1}]$ -modules

they are finitely generated.

Thus $P(K, t)$ and $P(L, t)$ have the stated form

$$0 \rightarrow K \rightarrow M \xrightarrow{\chi_s} M \rightarrow L \rightarrow 0$$

Also for each degree n .

$$\chi(K_n) - \chi(M_n) + \chi(M_{n+k_s}) - \chi(L_{n+k_s}) = 0$$

Multiply by t^{n+k_s} and sum

$\left(\sum_{n \geq 0}\right)$ we get

$$t^{k_s} P(K, t) - t^{k_s} P(M, t) + P(M, t)$$

$$- P(L, t) = g(t) \text{ for some polynomial } g(t)$$

Rearrange:

$$(1 - t^{k_s}) P(M, t) = P(L, t) - t^{k_s} P(K, t) + g(t)$$

$$= \frac{\text{polynomial}}{\prod_{i=1}^{s-1} (1 - t^{k_i})}$$

Therefore

$$P(M, t) = \frac{\text{polynomial}}{\prod_{i=1}^s (1 - t^{k_i})} \quad \square$$

Did we really need Hilbert and Serre to prove this?

Is it obvious why we need the polynomial g ?

Corollary. A is a Noetherian graded ring generated as an A_0 -algebra by homogeneous elements of degrees k_i . If each $k_i = 1$ then, for sufficiently large n , $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree $d-1$, where d is the order of the pole of $P(M, t)$ at $t = 1$. $d = d(M)$.

Proof. Here $\lambda(M_n) =$ coefficient of t^n in function $\frac{f(t)}{(1-t)^s}$

$$\approx \frac{f_1(t)}{(1-t)^d}$$

where f, f_1 are polynomials
 $f = f_1 (1-t)^{s-d}$
 Write $f_1(t) = \sum_{k=0}^N a_k t^k$

Also $\frac{1}{(1-t)^d} = \sum_{u=0}^{\infty} \binom{d+u-1}{d-1} t^u$

$$\frac{f_1(t)}{(1-t)^d} = \left(\sum_{k=0}^N a_k t^k \right) \left(\sum_{u=0}^{\infty} \binom{d+u-1}{d-1} t^u \right)$$

has, for $n \geq N$, coeff of t^n

$$\sum_{k=0}^n a_k \binom{d+(n-k)-1}{d-1}$$

This is a polynomial in n with leading term $\left(\sum_{k=0}^N a_k \right) \frac{n^{d-1}}{(d-1)!} \neq 0$

□

Discussion: Is anything about that at all remarkable?

Definition. The polynomial just described is the Hilbert function (or polynomial) of M .

Pre-class Warm-up!!

Let d be the order of the pole of

$$\frac{1+t+t^2}{(1-t)^d} = \sum_{n \geq 0} a_n t^n$$

at $t=1$. Which of the following correctly describes the degree of polynomial growth of the coefficients a_n as n increases?

A $d-1$ ✓

B d

C $d+1$

D None of the above.

$$\frac{1}{1-t} = 1 + t + t^2 + \dots$$

$a_n = 1$ always

polynomial of degree 0

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + \dots$$
$$= \sum_{n \geq 0} \binom{n+1}{1} t^n$$

$a_n = n+1$ polynomial of degree 1.

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + t^6 + \dots$$

a_n sequence is $1, 0, 1, 0, 1, 0, \dots$
not polynomial. This is described by
two polynomials $f_0(n) = 1, f_1(n) = 0$

If $n \equiv i \pmod{2}$ then $a_n = f_i(n)$

These a_n are (almost) Polynomial
On Residue Classes
= (almost) PORC

Write $d(M)$ for the order of the pole of $P(M, t)$ at 1. M is a f.g. graded A -module
 A is always Noetherian. A is graded and

Corollary. If a homogeneous element x in A is not a zero divisor on M then $d(M/xM) = d(M) - 1$.

Not a zero divisor means $xm = 0$ implies $m = 0$. This happens e.g. if $A = M$ is a domain and $x \neq 0$.

Proof. We had an exact sequence

$$0 \rightarrow K \rightarrow M \xrightarrow{x} M \rightarrow L \rightarrow 0,$$

x is not a zero divisor $\Leftrightarrow K = 0$,

$L = M/xM$, also.

$$\text{S.e.s. } 0 \rightarrow M_n \rightarrow M_{n+r} \rightarrow L_{n+r} \rightarrow 0$$

$r = \text{degree of } x$. Multiply by t^{n+1} , sum
 $t^r P(M, t) - P(M, t) + P(L, t) = g(t)$

$$P(L, t) = (1 - t^r) P(M, t) + g(t)$$

$$d(L) = d(M) - 1 \quad \square$$

There is a more subtle version of this in the context of local rings where we remove the assumption that x is homogeneous.

Examples.

$= A$ graded as usual.

$$P(A, t) = \frac{1}{(1-t)^3}.$$

$$1 + \frac{t^2}{1-t} = \frac{1-t+t^2}{1-t}$$

$$2. A = k[t^2, t^3] = k \oplus 0 \oplus kt^2 \oplus kt^3 \oplus \dots$$

$P(A, t)$

What is the Poincaré series of A (with respect to the k -dimension of terms)?

A. $\frac{1+t^2}{1-t}$

B. $\frac{1-t+t^2}{1-t} = \frac{1+t^3}{1-t^2}$

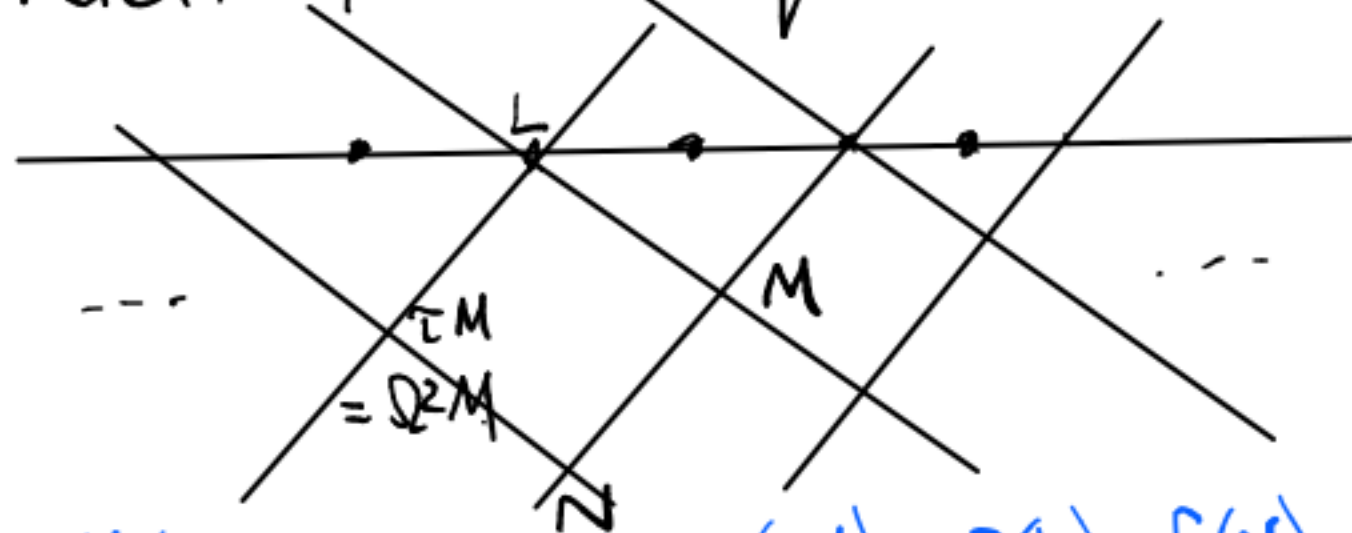
$$\frac{1+t+t^2}{1-t} = \frac{1-t^3}{(1-t)^2}$$

D None of the above.

The Auslander-Reiten quiver of a finite group. Math. Z. 1982

Theorem Let k a field, G a finite group. The tree class of ^{any comp of} the AR quiver of kG is an (extended) Dynkin diagram or one of 5 infinite trees.

Sketch. The AR quiver is



additive means $f(M) + f(\hat{\tau}M) = f(L) + f(N)$

$\Omega^i M = \text{kernel } K_{i-1}$ in a minimal projective resolution of M .

$\dim \Omega^{2i} M$ is an almost PRC function of i .

This comes about because $H^*(G, k)$ is a finitely generated graded-commutative ring.

(Evens-Venkov)

$H^*(G, M)$ is a finitely generated graded $H^*(G, k)$ -module.

$\dim(\Omega^i M)$ is determined by $H^*(G, M)$.

$M \mapsto f(M) =$ leading coefficient of the PRC polynomial. f is a periodic additive function on the AR-quiver. Happel-Preiser-Ringel showed this \Rightarrow the tree class is so.

Pre-class Warm-up!!

Netherian

Let J be an ideal of a commutative ring R .
Consider the two statements:

1. If R/J is Artinian then J is primary for some maximal ideal of R .
2. If J is primary for some maximal ideal of R then R/J is Artinian.

Which are true?

- A 1. is true. $R = k \times k, J = (0)$
- B 2. is true.
- C Both 1. and 2. are true.
- D Neither are true.

Artinian: DCC on submodules
 \Rightarrow f.g. modules have composition series

Fin. dim algebras over a field are Artinian

primary = one associated prime only

Fact: minimal primes containing J are associated primes.

Conclude (from J -primary, \mathcal{M} max)

\mathcal{M} is the unique minimal prime.

$= \sqrt{J}$, so $\mathcal{M}^n \subseteq J$ for some

$n \Rightarrow R/J$ is Artinian (it has a compn series).

The graded ring associated to an ideal

Proposition. *commutative*

Let J be an ideal of a Noetherian ring A .
Then the graded ring

$$G(A) = \bigoplus_{n=0}^{\infty} J^n / J^{n+1}$$

is Noetherian, generated by elements of degree 1. *is an A/J -algebra*.

If M is a finitely generated A -module then

$$G(M) = \bigoplus_{n=0}^{\infty} J^n M / J^{n+1} M$$

is a finitely generated $G(A)$ -module.

Proof A is Noetherian so J is a finitely generated ideal $J = (x_1, \dots, x_s)$

Let $\bar{x}_i = \text{image of } x_i \text{ in } J/J^2$
 $= x_i + J^2$.

Then $G(A) = A/J[\bar{x}_1, \dots, \bar{x}_s]$.

Why?

Also A/J is Noetherian, so $G(A)$ is Noetherian by Hilbert's basis theorem.

If M is a f.g. A -module then M/JM is a f.g. A/J -module and $J^n M / J^{n+1} M = (J^n / J^{n+1}) \cdot M/JM$ so M/JM generates $G(M)$ as a $G(A)$ -module. \square

*Why: we show $J^n / J^{n+1} \subseteq \text{right side}$
 $\forall n$. If $u \in J^n$ then $u + J^{n+1}$ is an A/J combination of products of the $\bar{x}_1, \dots, \bar{x}_s$.*

The same is true for $G(M)$ if it is defined by a filtration that eventually is multiplication by J and has $J M_i$ contained in M_{i+1} . Such is called J -stable.

We look for a situation where J is an ideal of A for which there is a suitable additive function on A/J -modules.

If J is primary for some maximal ideal of A then A/J is Artinian. *Take composition length over A/J as our additive function*
 Proposition.

Let J be an ideal of A so that A/J is Artinian, let M be a finitely generated A -module. Then

- $M/J^n M$ is of finite length for each $n \geq 0$.
- For all sufficiently large n this length is a polynomial $g(n)$ of degree $\leq s$ in n where s is the least number of generators of J .

Proof. We have seen: each $J^n M/J^{n+1} M$ is finitely generated as an A/J -module so has finite length, given by a Hilbert polynomial for $n \gg 0$.

The length of $M/J^n M$ is

$$\sum_{t=0}^{n-1} \text{length}(J^t M/J^{t+1} M)$$

$$= \sum_{t=0}^{n-1} H(t) + \text{constant}, \quad n \gg 0$$
 where H is polynomial of degree $\leq s-1$.

Thus length $(M/J^n M)$ is polynomial of degree $\leq s$ for $n \gg 0$. \square

The Proposition works for a filtration of M that is J -stable. Part c. says the degree and leading coefficient of $g(n)$ do not depend on the filtration chosen.

This is because

$$M_\nu \supseteq J^i M \supseteq M_{\nu+i} = J^i M_\nu$$
 where $M_i = J M_{i-1}$ if $i \geq N$.

Pre-class Warm-up!!

Let R be Noetherian

Let \mathcal{M} be a maximal ideal and J some ideal of a ring R .

Which of the following are true:

1. If J contains some power of \mathcal{M} then J is \mathcal{M} -primary.
2. If J is \mathcal{M} -primary then J contains some power of \mathcal{M} .

A Only 1. is true.

B Only 2. is true

C Both 1. and 2. are true ✓

D Neither is true.

J \mathcal{M} -primary \iff
 $\mathcal{M} = \text{unique prime containing } J$.
 $\implies \mathcal{M} = \sqrt{J}$
 $\implies \mathcal{M}^n \subseteq J$
 $\implies \mathcal{M} \subseteq \sqrt{J}$
 $\implies \mathcal{M} = \sqrt{J}$
 $\implies \mathcal{M} = \text{unique associated prime for } J$.

Definition. A is a Noetherian ring
 J is \mathfrak{M} -primary
 Aliyah-Macdonald and Matsumura write
 the polynomial

$$g(n) = \chi_J^M(n) = \text{length}(M/J^n M) \quad n \gg 0$$

for a finitely generated A -module M .

Matsumura calls it the Samuel function.

When $M = A$, Aliyah-Macdonald call
 it the characteristic polynomial of the
 ideal J .

Corollary.

For large n , $\text{length}(A/J^n)$ is a polynomial
 of degree $\leq s$, where $s =$ least number of

Proof Take $M = A$.

Proposition.

If J is \mathfrak{M} -primary where \mathfrak{M} is a maximal
 ideal then

$$\deg \chi_J^M(n) = \deg \chi_{\mathfrak{M}}^M(n)$$

Proof. We know $\mathfrak{M} \supseteq J \supseteq \mathfrak{M}^r$

for some r . Thus

$$\mathfrak{M}^n M \supseteq J^n M \supseteq \mathfrak{M}^{rn} M, \text{ so}$$

$$\chi_{\mathfrak{M}}^M(n) \leq \chi_J^M(n) \leq \chi_{\mathfrak{M}}^M(rn) \quad \forall n \gg 0$$

$$\deg \chi_{\mathfrak{M}}^M \leq \deg \chi_J^M \leq \deg \chi_{\mathfrak{M}}^M$$

Definition. $d(A)$ is

$$\deg \chi_{\mathfrak{M}}^A$$

Let A be a (Noetherian) ring. We already know that the Krull dimension $\dim A$

is the largest r so that there is a chain of prime ideals

$$\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \dots \subsetneq \mathfrak{P}_r \subset A.$$

e.g. $\dim \mathbb{Z} = 1$

We have seen:

$\dim A = 0 \iff A$ is Artinian.

Dimension is preserved under integral

This is a consequence of 'going up' and 'lying over'.

Goal: Let A be a Noetherian local ring with maximal ideal \mathfrak{M}

Let $\partial(A)$ = least number of generators of an \mathfrak{M} -primary ideal of A .

We indicate a proof that

$$\partial(A) = d(A) = \dim(A)$$

In general, Noetherian rings A need not have a Krull dimension.

Do maximal chains of prime ideals all have the same length, in some circumstances.

Question:

Why should we even be interested in knowing about $\partial(A)$, $d(A)$ or $\dim A$?

Are we interested in knowing about $\partial(A)$, $d(A)$ or $\dim A$?

Goal: Let A be a Noetherian local ring with maximal ideal \mathcal{M}

Let $\partial(A)$ = least number of generators of an \mathcal{M} -primary ideal of A .

We indicate a proof that

$$\partial(A) = d(A) = \dim(A)$$

showing $\partial(A) \geq d(A) \geq \dim A \geq \partial(A)$

Proposition.

Proof. We saw: if \mathcal{J} is \mathcal{M} -primary then # gens of $\mathcal{J} \geq d(A)$

Thus $\partial A \geq dA$. \square

Proposition 11.8 of Atiyah and Macdonald.

Notation as before.

Let M be a finitely-generated A -module, x in A a non-zero-divisor on M and $M' = M/xM$.

Then

$$\deg \chi_J^{M'} \leq \deg \chi_J^M - 1$$

We did this before in a graded situation

Proof. Let $N = xM$, which is isomorphic to M .

We have a s.e.s.

$$0 \rightarrow N/(N \cap J^n M) \rightarrow M/J^n M \rightarrow M'/J^n M' \rightarrow 0$$

Writing $g(n) = \text{length of } N/(N \cap J^n M)$ we have

$$g(n) - \chi_J^M(n) + \chi_J^{M'}(n) = 0 \quad \text{For } n \gg 0$$

Artin-Rees implies that $(N \cap J^n M)$ is a stable J -filtration of N , so $g(n)$ and $\chi_J^M(n)$ have the same leading term. Hence the result.

Corollary 11.9 of A and M.

If A is a Noetherian local ring and x is a non-zero-divisor in A , then

$$d(A/(x)) \leq d(A) - 1.$$

Proof Take $M = A$, so $M' = A/(x)$.
 \square

$$0 \rightarrow M \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$$

\parallel
 M'

is a s.e.s.

Proposition 11.10 $d(A) \geq \dim A$

Can you remember what $d(A)$ is? What did the last result say?

Proof. Induction on $d = d(A)$.

If $d = 0$ then $\text{Length}(A/\mathfrak{M}^n)$ is constant for all large n , so $\mathfrak{M}^n = \mathfrak{M}^{n+1}$ for some n , hence $\mathfrak{M}^n = 0$ by Nakayama's lemma. Thus A is an Artinian ring and $\dim A = 0$. Each $\mathfrak{M}^r/\mathfrak{M}^{r+1}$ is a fin. gen. A/\mathfrak{M} -module, so A has finite length.

Suppose $d > 0$ and the result for smaller values. Let $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \dots \subset \mathfrak{P}_r$

be any chain of prime ideals in A .

Let $x \in \mathfrak{P}_1$, $x \notin \mathfrak{P}_0$, $A' = A/\mathfrak{P}_0$.

Let x' be the image of x in A' .

Then $x' \neq 0$, and A' is an integral domain, so by 11.9

$$d(A' / (x')) \leq d(A') - 1$$

Also, if \mathfrak{M}' is the maximal ideal of A' , A'/\mathfrak{M}' is a homomorphism image of A/\mathfrak{M} hence $\text{Length}(A/\mathfrak{M}) \geq \text{Length}(A'/\mathfrak{M}')$ and therefore $d(A) \geq d(A')$. Thus

$$d(A' / (x')) \leq d(A) - 1 = d - 1.$$

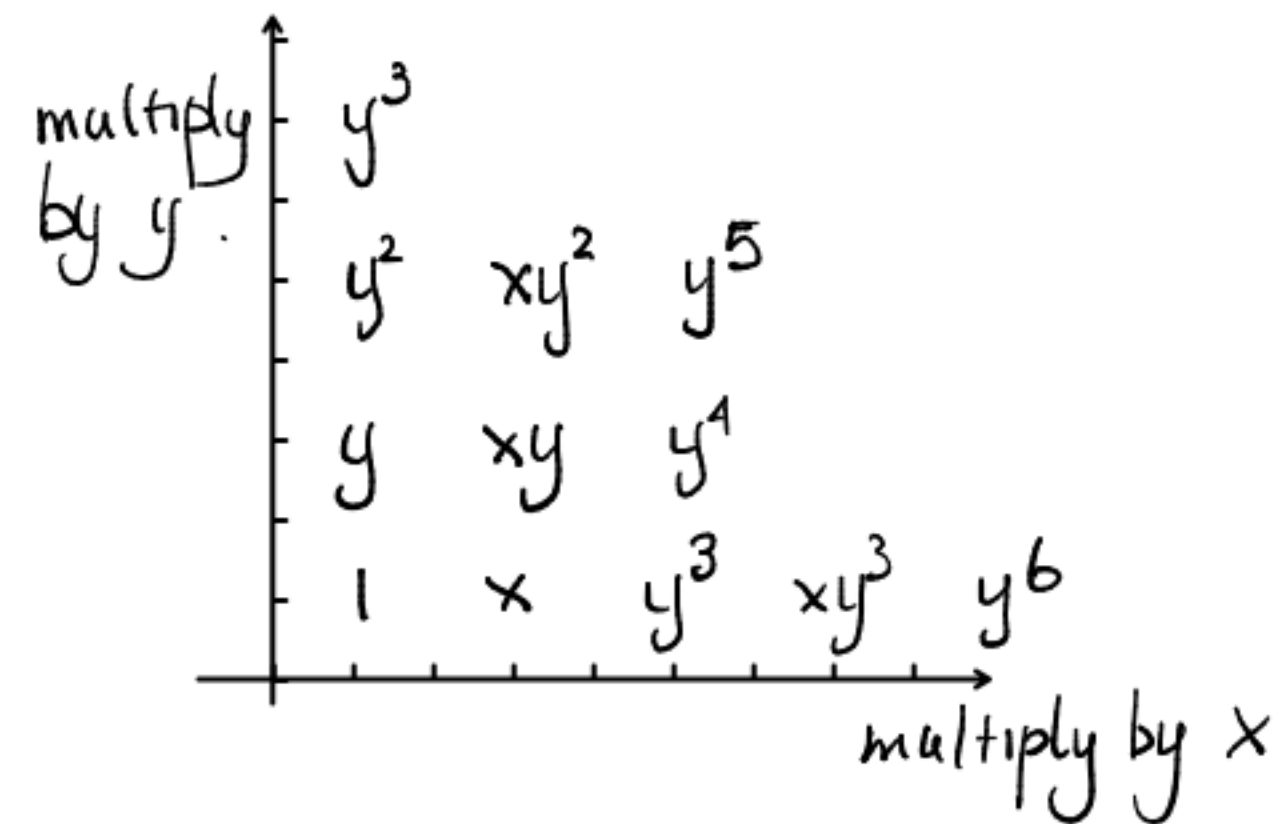
By induction, the length of any chain of prime ideals in $A' / (x')$ is $\leq d - 1$.

The images of $\mathfrak{P}_1, \dots, \mathfrak{P}_r$ in $A' / (x')$ form a chain of length $r - 1$, hence $r - 1 \leq d - 1$ and consequently $r \leq d$. Hence $\dim A \leq d$.

The final step $\dim A \geq d(A)$ in proving $d(A) = \dim A$ is a little technical and we are going to miss it out. The next corollaries do not depend on it.

The final step $\dim A \geq \partial(A)$ in proving $\partial(A) = d(A) = \dim A$ is a little technical and we are going to miss it out. The next corollaries do not depend on it.

However: 'proof by example' that $\dim A \geq \partial(A)$!



Recall: $\partial(A) \geq d(A) \geq \dim A \geq \partial(A)$

Corollary 1.11 of A&M

If A is Noetherian local ring then $\dim A$ is finite.

Definition.

The height of a prime ideal

It equals

Corollary 11.12 of A&M.

In a Noetherian ring every prime ideal has finite height. The set of prime ideals in a Noetherian ring satisfies DCC.

Corollary 11.15 of A&M

In a Noetherian local ring with maximal ideal \mathfrak{m} we have

$$\dim A \leq \dim (\mathfrak{m}/\mathfrak{m}^2)$$