Dimension theory and related things

1. Hilbert polynomials, Hilbert series, Poincare series

Definition. See the book of Atiyah – Macdonald.

A graded ring is a ring A together with a family $(A\_n)_{n\geq 0}$ of subgroups of the additive group of A, such that

1. $A = \bigoplus_{n \geq 0} A_n$

2. $A_m A_n \subseteq A_{m+n}$

We see: each A_n is an A_0-module and A_0 is a subring.

Examples. 1. $A = k[x_1, \dots, x_r]$, A_n = set of homogeneous polynomials of degree n.

$$A = k[t^2, t^3] \subseteq k[t]$$

$A_0 = k$, $A_1 = 0$, $A_n = k \ \forall n \geq 2$

3. If R is a ring, I is an ideal, put

$$A^* = \bigoplus_{n \geq 0} I^n, \quad A_n = I^n$$

$$R = I^0 = A_0.$$

4. The graded object associated to $I^j \subseteq I^{j-1} \subseteq \cdots R$ we get

$$G(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}$$

If $x_m \in I^m$, $x_n \in I^n$, write $\overline{x_m}$ = image in I^m / I^{m+1}, define $\overline{x_m} \cdot \overline{x_n} = \overline{x_m x_n}$

This is well-defined.

5 We could grade by a different monoid.

Pre-class Warm-up!!

Are you familiar with the formula for the dimension of the space of homogeneous polynomials in k[x_1, ... , x_d] of degree n as

$$\binom{n+d-1}{d-1} \quad ?$$

A Yes

B No

Definition.

Let A be a graded ring

A graded A-module is an A-module M together with a family $(M\_n)_{n\geq 0}$ of subgroups of M such that

1. $M = \bigoplus_{n\geq 0} M_n$

2. $A_m M_n \subseteq M_{m+n}$

An element $u \in M_m$ is called homogenous of degree m.

The subgroups M_m are the 'homogeneous components'

$$A_+ = \bigoplus_{n\geq 1} A_n$$ is an ideal of A.

More definitions:

Homogeneous elements, degree, homogeneous components, homomorphism of graded modules.

A_+

A homomorphism $\phi : L \to M$ is a homom. of graded modules if $\phi(L_m) \subseteq M_m$ $\forall m$.

Let A be commutative.

Proposition. TFAE for a graded ring A:

a. A is a Noetherian ring;

b. A_0 is Noetherian and A is finitely generated as an A_0-algebra.

Proof.

b => a is Hilbert's basis theorem. $A_0[u_1, \dots, u_s]$ is Noetherian, and A is an image of this.

a => b $A_0 = A/A_+$ is Noetherian.

The ideal A_+ is finitely generated, say by $x_1, \dots, x_s$. We may take these elements to be homogeneous. Why?

Let A' be the subring of A generated by $x_1, \dots, x_s$. and A_0 = A_0-subalgebra gen'd by $x_1, \dots, x_s$

We show that $A_n \subseteq A'$ for all $n \geq 0$.

Induction on n.

✓

For $n > 0$ let y be in A_n. Because y is in A_+ we can write y as a linear combination of the x_i, say

$$y = \sum_{i=1}^{s} a_i x_i$$

where $a_i \in A_{n-k_i}$

Let k_i be the degree of the homogeneous element x_i.

Each $k_i > 0$ so by induction each a_i is a polynomial in the x's with coefficients in A_0.

The same is true of y, therefore y is in A'.

Hence A_n is contained in A', so $A = A'$.

Hilbert functions

Let $A = \bigoplus_{n \geq 0} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring, and A is generated (as an A_0-algebra) by elements $x_1, \dots, x_s$ which we may choose to be homogeneous, of degrees k_1, ... , k_s

Let M be a finitely generated graded A-module, generated by homogeneous elements m_j, 1 ≤ j ≤ t. Each graded component M_n is now finitely generated as an A_0-module because M_n is generated as an A_0-module by elements $g_j(x) m_j$ where $g_j(x)$ is a monomial in the x_i of total degree $n - \deg m_j$.

Example $A = M = k[x_1, \dots x_n]$

$$P(M,t) = \frac{1}{(1-t)^n}$$

Let λ: fin. gen A_0-modules $\to \mathbb{Z}$ be an 'additive function', meaning $\forall$ s.e.s. of A_0-modules $0 \to L \to M \to N \to 0$ we have $\lambda(M) = \lambda(L) + \lambda(N)$.

e.g. if $A_0 = k$ is a field $\lambda = \dim$ is possible. Or λ = composition length.

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n \geq 0} \lambda(M_n)\, t^n \quad \text{in } Z[[t]].$$

Pre-class Warm-up!

Is the following true/false, obvious/
not obvious ?

Let M be an R-module where R is a commutative ring, and let r be an element of R.

There is an exact sequence

$$0 \to K \to M \xrightarrow{r} M \to L \to 0$$

where the middle map is multiplication by r and both K and L are annihilated by r.

A false

B (probably) true and not obvious

$K \cong \ker(r) \quad L \cong M/rM$.

r kills K and L

Definition. The Poincaré series of M (with respect to λ) is

$$P(M,t) = \sum_{n \geq 0} \lambda(M_n)\, t^n \quad \text{in } \mathbb{Z}[[t]].$$

Theorem (Hilbert, Serre)
Let A be a Noetherian graded ring, M a finitely generated graded A-module, λ a length function.
Then P(M,T) is a rational function in t of the form

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s} (1-t^{k_i})}, \quad f \in \mathbb{Z}[t]$$

Here $A = A_0[x_1, \dots, x_s]$, $\deg x_i = k_i$

Proof. Induction on s.
When $s=0$, $A = A_0$ and M is only non-zero in finitely many degrees. $P(M,t)$ is a polynomial.
Now suppose $s>0$ and result is true for smaller values. Consider the exact sequence

$$0 \to K \to M \xrightarrow{x_s} M \to L \to 0$$

x_s is supposed to be homogeneous so K and L are graded A-modules, finitely gen'd because A is Noetherian. They are killed by x_s. As $A_0[x_1, \dots, x_{s-1}]$-modules they are finitely generated.
Thus $P(K,t)$ and $P(L,t)$ have the stated form

$$0 \to K \to M \xrightarrow{x_s} M \to L \to 0.$$

Also for each degree n.

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0.$$

Multiply by t^{n+k_s} and sum $\left(\sum_{n\geq 0}\right)$ we get

$$t^{k_s}P(K,t) - t^{k_s}P(M,t) + P(M,t) - P(L,t) = g(t)$$

for some polynomial $g(t)$

Rearrange:

$$(1-t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t).$$

$$= \frac{\text{polynomial}}{\prod_{i=1}^{s-1}(1-t^{k_i})}$$

Therefore

$$P(M,t) = \frac{\text{polynomial}}{\prod_{i=1}^{s}(1-t^{k_i})}$$

$\square$

Did we really need Hilbert and Serre to prove this?

Is it obvious why we need the polynomial g?

Corollary. A is a Noetherian graded ring generated as an A_0-algebra by homogeneous elements of degrees k_i. If each k_i = 1 then, for sufficiently large n, $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree d-1, where d is the order of the pole of P(M,t) at t = 1. $d = d(M)$.

Proof. Here $\lambda(M_n)$ = coefficient of t^n in function $\frac{f(t)}{(1-t)^s}$

$$= \frac{f_1(t)}{(1-t)^d}$$

where f, f_1 are polynomials

$$f = f_1 (1-t)^{s-d}$$

Write $f_1(t) = \sum_{k=0}^{N} a_k t^k$

Also $\frac{1}{(1-t)^d} = \sum_{u=0}^{\infty} \binom{d+u-1}{d-1} t^u$

$$\frac{f_1(t)}{(1-t)^d} = \left(\sum_{k=0}^{N} a_k t^k\right)\left(\sum \binom{d+u-1}{d-1} t^u\right)$$

has, for $n \geq N$, coeff of t^n

$$\sum_{k=0}^{N} a_k \binom{d+(n-k)-1}{d-1}.$$

This is a polynomial in n with leading term $\left(\sum_{k=0}^{N} a_k\right)\frac{n^{d-1}}{(d-1)!} \neq 0$

$\square$

Discussion: Is anything about that at all remarkable?

Definition. The polynomial just described is the Hilbert function (or polynomial) of M.

Pre-class Warm-up!!

Let d be the order of the pole of

$$\frac{1+t+t^2}{(1-t)^d} = \sum_{n\geq 0} a_n t^n$$

at $t = 1$. Which of the following correctly describes the degree of polynomial growth of the coefficients a_n as n increases?

A d - 1 ✓

B d

C d + 1

D None of the above.

$$\frac{1}{1-t} = 1+t+t^2+\cdots$$

$a_n = 1$ always

polynomial of degree 0.

$$\frac{1}{(1-t)^2} = 1+2t+3t^2+\cdots$$

$$= \sum_{n\geq 0}\binom{n+1}{1} t^n.$$

$a_n = n+1$ polynomial of degree 1.

$$\frac{1}{1-t^2} = 1+t^2+t^4+t^6+\cdots$$

a_n sequence is $1,0,1,0,1,0,\ldots$

not polynomial. This is described by two polynomials $f_0(n) = 1$, $f_1(n) = 0$

If $n \equiv i \pmod{2}$ then $a_n = f_i(n)$

These a_n are (almost) Polynomial On Residue Classes

= (almost) PORC

Write d(M) for the order of the pole of P(M,t) at 1. M is a f g graded A-module

A is always Noetherian. A is graded now

Corollary. If a homogeneous element x in A is not a zero divisor on M then d(M/xM) = d(M) - 1.

Not a zero divisor means xm = 0 implies m = 0. This happens e.g. if A = M is a domain and x ≠ 0.

Proof. We had an exact sequence

$$0 \to K \to M \xrightarrow{x} M \to L \to 0,$$

x is not a zero divisor $\Leftrightarrow K = 0$,

$L = M/xM$, also.

s.e.s. $0 \to M_n \to M_{n+r} \to L_{n+r} \to 0$

$r =$ degree of x. Multiply by t^{n+1}, sum

$$t^r P(M,t) - P(M,t) + P(L,t) = g(t)$$

$$P(L,t) = (1-t^r)P(M,t) + g(t)$$

$$d(L) = d(M) - 1$$

□

There is a more subtle version of this in the context of local rings where we remove the assumption that x is homogeneous.

Examples.

$= A$ graded as usual.

$$P(A,t) = \frac{1}{(1-t)^s}.$$

$$1 + \frac{t^2}{1-t} = \frac{1-t+t^2}{1-t}$$

2. A = k[t^2, t^3] $= k \oplus 0 \oplus k t^2 \oplus k \cdot t^3 \oplus \ldots$

What is the Poincaré series of A $P(A,t)$ (with respect to the k-dimension of terms)?

A. $\frac{1+t^2}{1-t}$

B. $\frac{1-t+t^2}{1-t} = \frac{1+t^3}{1-t^2}$

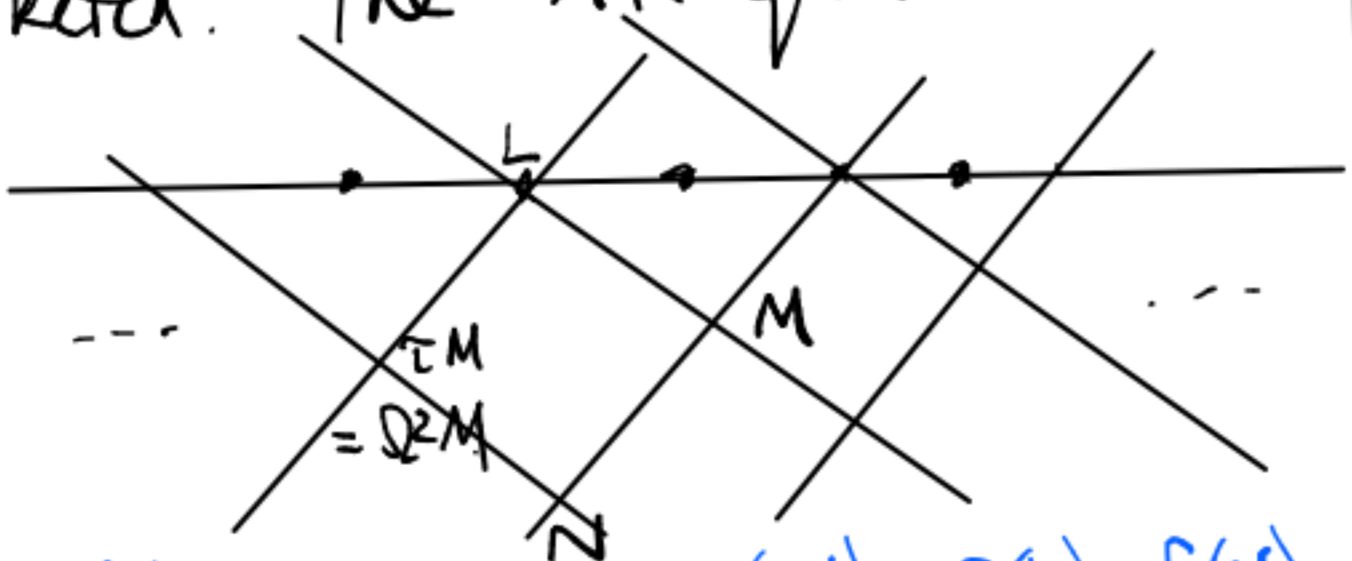
$\frac{1+t+t^2}{1-t} = \frac{1-t^3}{(1-t)^2}$

D None of the above.

The Auslander-Reiten quiver of a finite group. Math. Z. 1982

Theorem k a field, G a finite group. The tree class of any cmpt of the AR quiver of kG is an (extended) Dynkin diagram or one of 5 infinite trees.

Sketch. The AR quiver is



additive means $f(M) + f(\tau M) = f(L) + f(N)$

$\Omega^i M$ = kernel K_{i-1} in a minimal projective resolution of M.

$\dim \Omega^{2i} M$ is an almost PORC function of i.

This comes about because $H^*(G,k)$ is a finitely generated graded-commutative ring (Evens-Venkov)

$H^*(G,M)$ is a finitely generated graded $H^*(G,k)$-module.

$\dim(\Omega^i M)$ is determined by $H^*(G,M)$.

$M \mapsto f(M) =$ leading coefficient of the PORC polynomial. is a periodic additive function on the AR-quiver. Happel-Preiser-Ringel showed this $\Rightarrow$ the tree class is so.

Pre-class Warm-up!!

Noetherian

Let J be an ideal of a commutative ring R.
Consider the two statements:

1. If R/J is Artinian then J is primary for some maximal ideal of R.

2. If J is primary for some maximal ideal of R then R/J is Artinian.

Which are true?

A 1. is true. × $R = k \times k$, $J = (0)$

B 2. is true. ✓

C Both 1. and 2. are true.

D Neither are true.

Artinian: DCC on submodules
⇒ f.g. modules have composition series

Fin. dim algebras over a field are Artinian

primary = one associated prime only

Fact: minimal primes containing J are associated primes.
Conclude (from J-primary, $\mathfrak{m}$ maxl) $\mathfrak{m}$ is the unique minimal prime $= \sqrt{J}$, so $\mathfrak{m}^n \subseteq J$ for some n ⇒ R/J is Artinian (it has a compn series).

The graded ring associated to an ideal

Proposition. commutative
Let J be an ideal of a Noetherian ring A.
Then the graded ring

$$G(A) = \bigoplus_{n=0}^{\infty} J^n / J^{n+1}$$

is Noetherian, generated by elements of degree 1. as an A/J-algebra.
If M is a finitely generated A-module then

$$G(M) = \bigoplus_{n=0}^{\infty} J^n M / J^{n+1} M$$

is a finitely generated G(A)-module.

Proof A is Noetherian so J is a finitely generated ideal $J = (x_1, \dots, x_s)$

Let $\bar{x}_i$ = image of x_i in J/J^2
$= x_i + J^2$.

Then $G(A) = A/J[\bar{x}_1, \dots, \bar{x}_s]$. Why?

Also A/J is Noetherian, so G(A) is Noetherian by Hilbert's basic theorem.

If M is a f.g. A-module then M/JM is a f.g. A/J-module and $J^n M / J^{n+1} M = (J^n/J^{n+1}) \cdot M/JM$

so M/JM generates G(M) as a G(A)-module. $\square$

Why: we show $J^n/J^{n+1} \subseteq$ right side $\forall n$. If $u \in J^n$ then $u + J^{n+1}$ is an A/J combination of products of the $\bar{x}_1, \dots, \bar{x}_s$.

The same is true for G(M) if it is defined by a filtration that eventually is multiplication by J and has J M_i contained in M_{i+1}. Such is called J-stable.

We look for a situation where J is an ideal of A for which there is a suitable additive function on A/J-modules.

If J is primary for some maximal ideal of A then A/J is Artinian. Take composition length over A/J as our 'additive function'

Proposition.
Let J be an ideal of A so that A/J is Artinian, let M be a finitely generated A-module. Then
a. M/J^nM is of finite length for each $n \geq 0$.
b. For all sufficiently large n this length is a polynomial g(n) of degree $\leq s$ in n where s is the least number of generators of J.

Proof. We have seen: each $J^nM/J^{n+1}M$ is finitely generated as an A/J-module so has finite length, given by a Hilbert polynomial for $n >> 0$.

The length of M/J^nM is

$$\sum_{t=0}^{n-1} \text{length}\left(J^tM/J^{t+1}M\right)$$

$$= \sum_{t=0}^{n-1} H(t) + \text{constant}, \quad n >> 0$$

where H is polynomial of degree $\leq s-1$

Thus length (M/J^nM) is polynomial of degree $\leq s$ for $n >> 0$. $\square$

The Proposition works for a filtration of M that is J-stable. Part c. says the degree and leading coefficient of g(n) do not depend on the filtration chosen.
This is because

$$M_i \supseteq J^iM \supseteq M_{N+i} = J^iM_N$$

where $M_i = JM_{i-1}$ if $i \geq N$.

Pre-class Warm-up!!

Let R be Noetherian

Let $\mathfrak{m}$ be a maximal ideal and J some ideal of a ring R.
Which of the following are true:

1. If J contains some power of $\mathfrak{m}$ then J is $\mathfrak{m}$-primary.

2. If J is $\mathfrak{m}$-primary then J contains some power of $\mathfrak{m}$

A Only 1. is true.

B Only 2. is true

C Both 1. and 2. are true ✓

D Neither is true.

J $\mathfrak{m}$-primary $\Longleftrightarrow$

$\mathfrak{m}$ = unique prime containing J.

$\Longrightarrow \mathfrak{m} = \sqrt{J}$

$\Longrightarrow \mathfrak{m}^r \subseteq J$

$\Longrightarrow \mathfrak{m} \subseteq \sqrt{J}$

$\Longrightarrow \mathfrak{m} = \sqrt{J}$

$\Longrightarrow \mathfrak{m}$ = unique associated prime for J.

Definition. A is a Noetherian ring
J is $\mathfrak{m}$-primary
Aliyah-Macdonald and Matsumura write the polynomial

g(n) = $\chi^M_J(n) = \text{length}(M/J^n M) \quad n \gg 0$

for a finitely generated A-module M.

Matsumura calls it the Samuel function. When M = A, Aliyah-Macdonald call it the characteristic polynomial of the ideal J.

Corollary.
For large n, length (A/J^n) is a polynomial of degree ≤ s, where s = least number of

Proof Take $M = A$.

Proposition.
If J is $\mathfrak{m}$-primary where $\mathfrak{m}$ is a maximal ideal then

$$\deg \chi^M_J(n) = \deg \chi^M_{\mathfrak{m}}(n)$$

Proof. We know $\mathfrak{m} \supseteq J \supseteq \mathfrak{m}^r$ for some r. Thus

$\mathfrak{m}^n M \supseteq J^n M \supseteq \mathfrak{m}^{rn} M$, so

$\chi^M_{\mathfrak{m}}(n) \leq \chi^M_J(n) \leq \chi^M_{\mathfrak{m}}(rn) \ \forall n \gg 0$

$\deg \chi^M_{\mathfrak{m}} \leq \deg \chi^M_J \leq \deg \chi^M_{\mathfrak{m}}$.

Definition. d(A) is

$\deg \chi^A_{\mathfrak{m}}$

Let A be a (Noetherian) ring. We already know that the Krull dimension dim A

is the largest r so that there is a chain of prime ideals

$\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_r \subset A$.

e.g. $\dim \mathbb{Z} = 1$

We have seen:

Dim A = 0 <=> A is Artinian.

Dimension is preserved under integral

This is a consequence of 'going up' and 'lying over'

Goal: Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$

Let $\partial(A)$ = least number of generators of an $\mathfrak{m}$-primary ideal of A.

We indicate a proof that

$\partial(A) = d(A) = \dim(A)$

In general, Noetherian rings A need not have a Krull dimension.

Do maximal chains of prime ideals all have the same length, in some circumstances.

Question:
Why should we even be interested in knowing about $\partial(A)$, $d(A)$ or $\dim A$?

Are we interested in knowing about $\partial(A)$, $d(A)$ or $\dim A$?

Goal: Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$

Let $\partial(A)$ = least number of generators of an $\mathfrak{m}$-primary ideal of A.

We indicate a proof that

$$\partial(A) = d(A) = \dim(A)$$

showing $\partial(A) \geq d(A) \geq \dim A \geq \partial(A)$

Proposition.

Proof. We saw: if J is $\mathfrak{m}$-primary then #gens of J $\geq d(A)$

Thus $\partial A \geq d A$. $\square$

Proposition 11.8 of Atiyah and Macdonald.
Notation as before.
Let M be a finitely-generated A-module, x in A a non-zero-divisor on M and $M' = M/xM$.
Then

$$\deg \chi_J^{M'} \leq \deg \chi_J^{M} - 1$$

We did this before in a graded situation

Proof. Let $N = xM$, which is isomorphic to M.
We have a s.e.s.

$$0 \to N/(N \cap J^nM) \to M/J^nM \to M'/J^nM' \to 0$$

Writing $g(n)$ = length of $N/(N \cap J^nM)$ we have

$$g(n) - \chi_J^M(n) + \chi_J^{M'}(n) = 0 \quad \text{For } n \gg 0$$

Artin-Rees implies that ~~$(N \cap J^nN)$~~ $(N \cap J^n)$ is a stable J-filtration of N, so $g(n)$ and $\chi_J^M(n)$ have the same leading term. Hence the result.

Corollary 11.9 of A and M.
If A is a Noetherian local ring and x is a non-zero-divisor in A, then

$$d(A/(x)) \leq d(A) - 1.$$

Proof Take $M = A$, so $M' = A/(x)$.

$\square$

$$0 \to M \xrightarrow{x} M \to \underset{\underset{\textstyle M'}{\shortparallel}}{M/xM} \to 0$$

is a s.e.s.

Pre-class Warm-up

Can we remember what d(A), $\partial(A)$ and dim A are?

A Yes

B No

For $\partial(A)$, $d(A)$, A is supposed to be a Noetherian local ring.

$d(A)$ = degree of χ

$\chi(n)$ = length of $A/\mathfrak{m}^n$, $n \gg 0$

$\mathfrak{m}$ = max ideal of A.

dim A = largest r such that $\exists$ chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$

$\partial(A)$ = least number of generators of an $\mathfrak{m}$-primary ideal of A.

$\partial(A) \geq d(A)$

We show $d(A) \geq \dim A$.

Proposition 11.10 $d(A) \geq \dim A$

Can you remember what $d(A)$ is? What did the last result say?

Proof. Induction on $d = d(A)$.

If $d = 0$ then Length $(A/\mathfrak{m}^n)$ is constant for all large n, so $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n, hence $\mathfrak{m}^n = 0$ by Nakayama's lemma. Thus A is an Artinian ring and $\dim A = 0$. Each $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is a fin. gen. $A/\mathfrak{m}$-module, so A has finite length.

Suppose $d > 0$ and the result for smaller values. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$

be any chain of prime ideals in A.

Let $x \in \mathfrak{p}_1$, $x \notin \mathfrak{p}_0$, $A' = A/\mathfrak{p}_0$.

Let x' be the image of x in A'.
Then $x' \neq 0$, and A' is an integral domain, so by 11.9

$d(A' / (x')) \leq d(A') - 1$

Also, if $\mathfrak{m}'$ Is the maximal ideal of A', $A'/\mathfrak{m}'^n$ is a homomorphism image of $A/\mathfrak{m}^n$ hence Length$(A/\mathfrak{m}^n) \geq$ Length $(A'/\mathfrak{m}'^n)$ and therefore $d(A) \geq d(A')$. Thus

$d(A' / (x')) \leq d(A) - 1 = d - 1$.
By induction, the length of any chain of prime ideals in $A' / (x')$ is $\leq d - 1$. The images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in $A' / (x')$ form a chain of length $r - 1$, hence $r - 1 \leq d - 1$ and consequently $r \leq d$. Hence $\dim A \leq d$.

Why are these images prime?

The final step $\dim A \geq \partial(A)$ in proving $\partial(A) = d(A) = \dim A$ is a little technical and we are going to miss it out. The next corollaries do not depend on it.

The final step $\dim A \geq \partial(A)$ in proving $\partial(A) = d(A) = \dim A$ is a little technical and we are going to miss it out. The next corollaries do not depend on it.

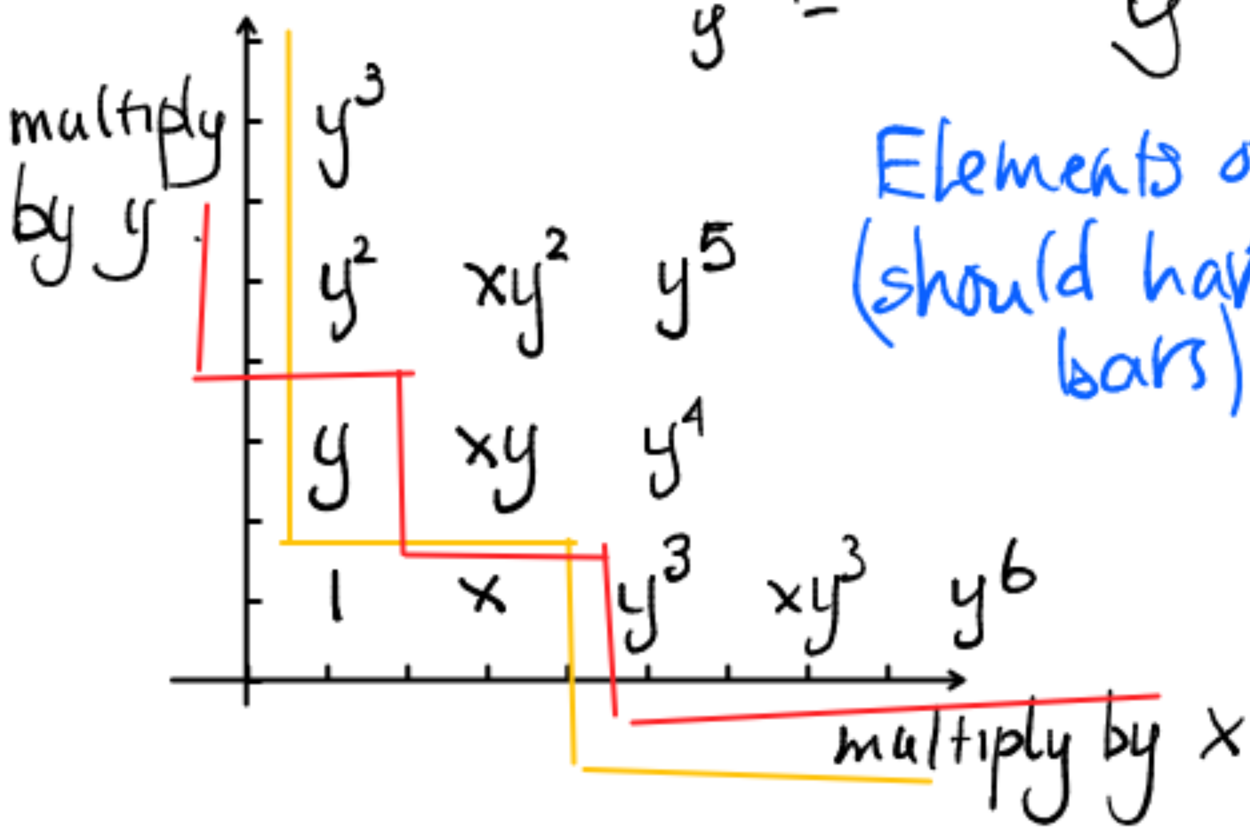
However: `proof by example' that $\dim A \geq \partial(A)$!

To prove this we construct an $\mathfrak{m}$-primary ideal generated by dim A elements.

Example. Consider $A = k[x,y]/(x^2-y^3)$

$\mathfrak{m} = (\bar{x}, \bar{y})$, $\bar{x}$ = image of x in A

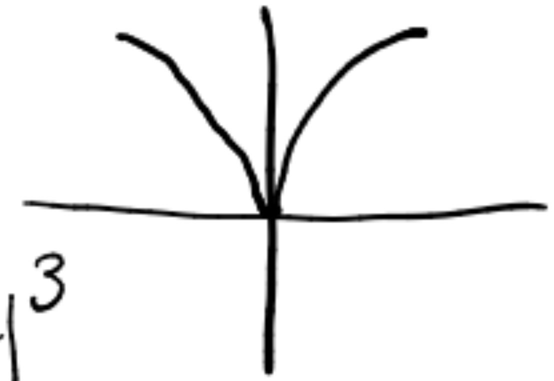
$\bar{y}$ = y



Elements of A (should have bars)

Consider $A_{\mathfrak{m}}$.

Picture



$x^2 = y^3$

Observe

$(\bar{y})$ yellow is an $\mathfrak{m}$-primary ideal because

$\mathfrak{m}^2 \subseteq (\bar{y})$.

red

Here $(\bar{y})$ has one generator

$\mathfrak{m}$ has two generators.

$\partial(A_{\mathfrak{m}}) = 1 \leq \dim A$. ✓

Recall: $\partial(A) \geq d(A) \geq \dim A \geq \partial(A)$

Corollary 1.11 of A&M
If A is Noetherian local ring then $\dim A$ is finite.

Definition.
The height of a prime ideal $\mathfrak{p}$ is the largest r such that $\exists$ chain of prime ideals $\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_r = \mathfrak{p}$ in A.
It equals $\dim A_{\mathfrak{p}}$.

of $\mathfrak{p}$ is the largest r so that there is a chain $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ of prime ideals. It equals $\dim A/\mathfrak{p}$.

Corollary 11.12 of A&M.
In a Noetherian ring every prime ideal has finite height. The set of prime ideals in a Noetherian ring satisfies DCC.

Corollary 11.15 of A&M
In a Noetherian local ring with maximal ideal $\mathfrak{m}$ we have
$\operatorname{Dim} A \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ where $k = A/\mathfrak{m}$.

Proof. # gens of $\mathfrak{m}$ = # gens of $\mathfrak{m}/\mathfrak{m}^2$ by Nakayama (if $\mathfrak{m}^2 \sqcup X$ generates $\mathfrak{m}$ then X generates $\mathfrak{m}$)

Definition. A Noetherian local ring A is regular if no of generators of $\mathfrak{m}$ = $\dim A$.
Always $\geq$

Corollary 11.16 Let A be a Noetherian ring, and $x_1, \ldots, x_r$ in A. Then every prime ideal minimal over $(x_1, \ldots, x_r)$ has height $\leq r$.

The case $r = 1$ plus a little bit is known as Krull's

Corollary 11.19. Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$

Then $\dim A = \dim \hat{A}_{\mathfrak{m}}$