The dimension of affine rings

Affine rings = finitely generated algebras over a field k.

It's nice to know

Theorem.
If k is a field then dim $k[x_1, \ldots ,x_r] = r$

Noether Normalization

If A is a graded algebra finitely generated over k then the degree of the Hilbert polynomial is dim A - 1

Lemma 13.2. in Eisenbud.

Let k be a field and f in T = k[x_1, … ,x_r] a non-constant polynomial.
There are elements x'_1, … , x'_{r-1} in T so that
T is a finitely generated module over the k-subalgebra generated by x'_1, … , x'_{r-1} and f. T is integral over S = subalg. gen'd by $x'_1, \dots, x'_{r-1}, f$

The x'_i can be chosen in various ways. We will show that we can choose x'_i of the form x_i - x_r^{e^i} for any sufficiently large e. $\in \mathbb{Z}$

If f is homogeneous, they can be chosen (in a different way) homogeneous.

Proof. Write f as a polynomial in $x_1', \dots, x_{r-1}', x_r$.
This means: where we see x_i we write $x_i' + x_r^{e^i}$
A monomial $x_1^{a_1} \cdots x_r^{a_r}$ becomes

$$(x_1' + x_r^{e^1})^{a_1} \cdots (x_{r-1}' + x_r^{e^{r-1}})^{a_{r-1}} x_r^{a_r}$$

$$= x_1'^{a_1} \cdots x_{r-1}'^{a_{r-1}} x_r^{a_r} + \cdots + x_r^{\overbrace{a_1e^1 + \cdots a_{r-1}e^{r-1} + a_r}^{= d}}$$

If $e \gg 0$ the last term has highest degree x_r^d.
For different monomials, the highest degrees d are distinct (the a_i are the digits in the e-adic expansion of d).
No cancellation occurs between them.
The very highest degree is a power of x_r.
x_r is a root of a monic polynomial over $k[x_1', \dots, x_{r-1}', f] = S$ $f(x_r) - f$
Thus T is generated over S by $1, x_r, x_r^2, \dots, x_r^{d-1}$. ∎