

The dimension of affine rings

Affine rings = finitely generated algebras over a field k .

It's nice to know

Theorem.

If k is a field then $\dim k[x_1, \dots, x_r] = r$

Noether Normalization

If A is a graded algebra finitely generated over k then the degree of the Hilbert polynomial is $\dim A - 1$

Lemma 13.2. *in Eisenbud.*

Let k be a field and f in $T = k[x_1, \dots, x_r]$ a non-constant polynomial.

There are elements x'_1, \dots, x'_{r-1} in T so that

T is a finitely generated module over the k -subalgebra generated by x'_1, \dots, x'_{r-1} and f .

T is integral over $S = \text{subalg. gen'd by } x'_1, \dots, x'_{r-1}, f$

The x'_i can be chosen in various ways. We will show that we can choose x'_i of the form $x_i - x_r^{e^i}$ for any sufficiently large $e \in \mathbb{Z}$.

If f is homogeneous, they can be chosen (in a different way) homogeneous.

Proof. Write f as a polynomial in $x'_1, \dots, x'_{r-1}, x_r$.

This means: where we see x_i we write

$$x'_i + x_r^{e^i}$$

A monomial $x_1^{a_1} \dots x_r^{a_r}$

becomes

$$(x'_1 + x_r^{e^1})^{a_1} \dots (x'_{r-1} + x_r^{e^{r-1}})^{a_{r-1}} x_r^{a_r}$$

$$= x_1^{a_1} \dots x_{r-1}^{a_{r-1}} x_r^{a_r} + \dots$$

$$+ x_r^{a_1 e^1 + \dots + a_{r-1} e^{r-1} + a_r}$$

$= d$

If $e \gg 0$ the last term has highest degree x_r^d . For different monomials, the highest degrees d are distinct (the a_i are the digits in the e -adic expansion of d).

No cancellation occurs between terms. The very highest degree is a power of x_r . x_r is a root of a monic polynomial over $k[x'_1, \dots, x'_{r-1}, f] = S$. Thus T is generated over S by $1, x_r, x_r^2, \dots, x_r^{d-1}$. \square

Pre-class Warm-up

Is the following statement obvious?

Let A be a commutative ring and let B be a homomorphism image of A .

Then $\dim A \leq \dim B$.

A Yes, it is obvious

B No, it is not obvious.

We have the understanding that if a statement is not true, then it is not obvious.

Today: we will show
 $\dim \mathbb{K}[x, y] / (x^2 - y^3) = 1$.

$\dim \mathbb{K}[x_1, \dots, x_d] = d$.

Theorem 13.1

Let k be a field. Then
 $\dim k[x_1, \dots, x_r] = r$.

Proof. Write $T = k[x_1, \dots, x_r]$.

Induction on r .

$r=0$: $T = k$ has dimension 0.

Suppose $r > 0$ and result holds for $r-1$.

$0 \subset (x_1) \subset (x_1, x_2) \subset \dots \subset (x_1, \dots, x_r)$
is a chain of prime ideals so
 $\dim T \leq r$.

Let $0 \subset P_1 \subset \dots \subset P_m$ be a chain
of prime ideals. Let $0 \neq f \in P_1$.

We can find x'_1, \dots, x'_{r-1} so that

T is integral over $S := k[x'_1, \dots, x'_{r-1}, f]$

By 'incomparability'

$0 \subset S \cap P_1 \subset \dots \subset S \cap P_m$ is a
chain of distinct prime ideals.

Factor out (f) to get a chain
of length $m-1$ in $S/(f)$:

$$\frac{S \cap P_1}{(f)} \subset \dots \subset \frac{S \cap P_m}{(f)}$$

$S/(f)$ is an image of $k[x'_1, \dots, x'_{r-1}]$
which has dimension $\leq r-1$, by induction
(image of a polynomial ring with $r-1$
variables).

Hence $m-1 \leq r-1$, $m \leq r$. \square

Corollary. Any k -algebra that can be
generated by r elements has dimension $\leq r$.

Example.

Proof. In $k[x, y]$ take $f = x^2 - y^3$

We can find x' so that

$k[x, y]$ is integral over $k[x', f]$

$A = k[x, y]/(x^2 - y^3)$ is

integral over $k[x', f]/(f)$

an image of
 ~~\cong~~ $k[x']$ which has $\dim 1$

Therefore $\dim A \leq 1$.

$\dim A \neq 0$.

$0 \subset (\bar{x}, \bar{y})$ is a chain
of prime ideals.

Simplified and weaker form of Noether's

see Atiyah-Macdonald

Theorem (compare 13.3 in Eisenbud)

Let k be a field and let $A \neq 0$ be a finitely generated k -algebra (an 'affine ring over k ').

There is a polynomial subring

$S = k[x_1, \dots, x_d]$ such that A is integral over S .

Proof. Let y_1, \dots, y_n generate A as a k -algebra.

Renumber them so that y_1, \dots, y_r are algebraically independent over k , each of y_{r+1}, \dots, y_n is algebraic over $k[y_1, \dots, y_r]$.

Induction on n . $n = r$: done.

Suppose $n > r$, and result holds for $n-1$ generators.

y_n is algebraic over $k[y_1, \dots, y_{n-1}]$
so \exists non-zero polynomial f so that
 $f(y_1, \dots, y_{n-1}, y_n) = 0$.

By lemma 13.2 $\exists x'_1, \dots, x'_{n-1}$ in A so that A is integral over $k[x'_1, \dots, x'_{n-1}, f] \subset k[x'_1, \dots, x'_{n-1}]$

By induction $k[x'_1, \dots, x'_{n-1}]$ is integral over a polynomial ring S .

Integral extensions remain integral. \square

Is it obvious?: $k[x_1, \dots, x_d]$ is a polynomial ring $\Leftrightarrow x_1, \dots, x_d$ are algebraically independent over k ?

Theorem (compare 13.3 in Eisenbud)

Let k be a field and let $A \neq 0$ be a finitely generated k -algebra (an 'affine ring over k ').

There is a polynomial subring

$S = k[x_1, \dots, x_d]$ such that A is integral over S .

Furthermore, if $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \dots \subset \mathcal{I}_m$

is a chain of ideals of A with $\dim A/\mathcal{I}_j = d_j$

and $d_1 > d_2 > \dots > d_m > 0$

then S can be chosen so that

$$\mathcal{I}_j \cap S \cong (x_{d_j}, \dots, x_d) \quad .$$

If the ideals are homogeneous, the x_j can be chosen to be homogeneous.

Pre-class Warm-up!!

Did you see a proof of Hilbert's basis theorem in a previous course?

A Yes

B No

Theorem. If A is an affine domain over a field k , then $\dim A = \text{tr.deg.}_k A$.

Proof. Let $S = k[x_1, \dots, x_d]$ be the polynomial subring of A over which A is integral. Then

$\text{tr.deg. } S = d$

and the field of fractions of A is a finite degree extension of S , so $\text{tr.deg. } A = d$ also.

the field of fractions $k(x_1, \dots, x_d)$ of

There is an extra statement to this which completes Theorem A of Eisenbud in section 13:

$\dim A$ equals the length of every maximal chain of primes in A .

Corollary 13.4 Every prime ideal \mathfrak{p} of an affine domain A satisfies
 $\text{height } \mathfrak{p} + \text{coheight } \mathfrak{p} = \dim A$.

$\text{tr.deg. } A_{\mathfrak{p}} = \dim A / \mathfrak{p}$

Corollary 13.7.

If R is a Noetherian graded ring with Hilbert polynomial $P_R(t)$ then
 $\dim R = 1 + \deg P_R(t)$.

$\dim R = \text{order of the pole at } t = -1 \text{ of Poincaré series of } R$.