Groebner basis theory

Books:
Eisenbud Chapter 15
Dummit and Foote Section 9.6

We have seen how effective it is to compute with monomial ideals of $S = k[x_1, \dots, x_n]$

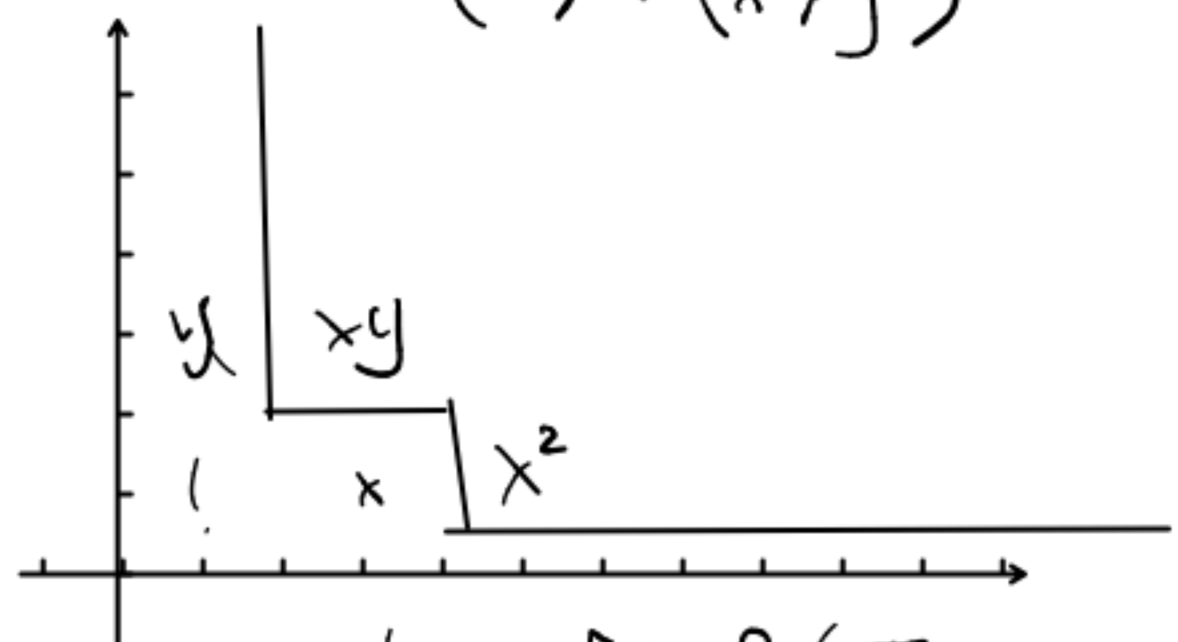
Definition. A <u>monomial</u> of S is a product $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = x^a$ where $a = (a_1, \dots, a_n) = \partial(x^a)$ is the <u>multidegree</u>.

Perhaps it is sometimes a scalar multiple?

A monomial ideal I is one generated by monomials.
It has a k-basis of monomials.

We have seen e.g. we can compute intersections of m. ideals.

$$(x^2, xy) = (x) \cap (x^2, xy, y^2)$$
$$= (x) \cap (x^2, y)$$



We see a basis for S/I e.g. $\bar{1}, \bar{x}, \bar{y}, \bar{y}^2, \dots$

We can easily compute the Hilbert function & Poincaré series of S/I.

We see that monomial ideals are finitely generated.

In fact we know ideals of S are all finitely gen'd.

Gordan's 1900 proof of Hilbert's basis theorem used this.

Proof of Hilbert's basis theorem

Definition. A basis for an ideal is a set of ideal generators ~~for~~ the ideal.

Hilbert's Basis Theorem.
If R is a Noetherian ring then so is the polynomial ring R[x].
Every ideal of R[x] has a finite basis.

Proof.
Let $I \subset R[x]$ be an ideal.
Let
$L = \{$leading coefficients of elements of $I\}$
Claim: this is an ideal of R.
(Proof $f = ax^d +$ lower $\in I$
$g = bx^e +$ lower $\in I$
then $ra - b$ is either 0 or the leading coeff of $rx^e f - x^d g$)

L is finitely generated by
$a_1, \ldots, a_n \in R$.
Let $f_i \in I$ have leading coeff a_i
Put $e_i = \deg f_i$, $N = \max\{e_1, \ldots, e_n\}$
If $0 \le d \le N-1$ put
$L_d = \{$leading coeffs of polys in I of degree $d\}$
This is also an ideal.
$L_d = (b_{d,1}, \ldots, b_{d,n_d})$. $b_{ij} \in R$
Find $f_{d,i} \in I$ of degree d
with leading coeff $b_{d,i}$.

Claim:

$I = (\{f_1, \ldots, f_n\} \cup \{f_{d,i} \mid 0 \le d < N, 1 \le i \le n_d\})$

(Pf. Let I' be the ideal on the right.

$I' \subseteq I$. If $\neq$, pick $f \in I - I'$ of least degree.

If $\deg f \geq N$ then its leading coeff is a combn of $a_1, \ldots, a_n$.

Let g = same combn of $x^{\deg f - \deg f_i} f_i \in I$

Now $f-g \in I - I'$ has smaller degree than f.

Contradiction.

Similar if $\deg f < N$. Find g = combn of $f_{\deg f, i} \in I'$ with same leading term as f. Now $f-g \in I - I'$ has smaller degree than f. contradiction. $\square$