Groebner basis theory

Books:
Eisenbud Chapter 15
Dummit and Foote Section 9.6

We have seen how effective it is to compute with monomial ideals of $S = k[x_1, \dots, x_n]$

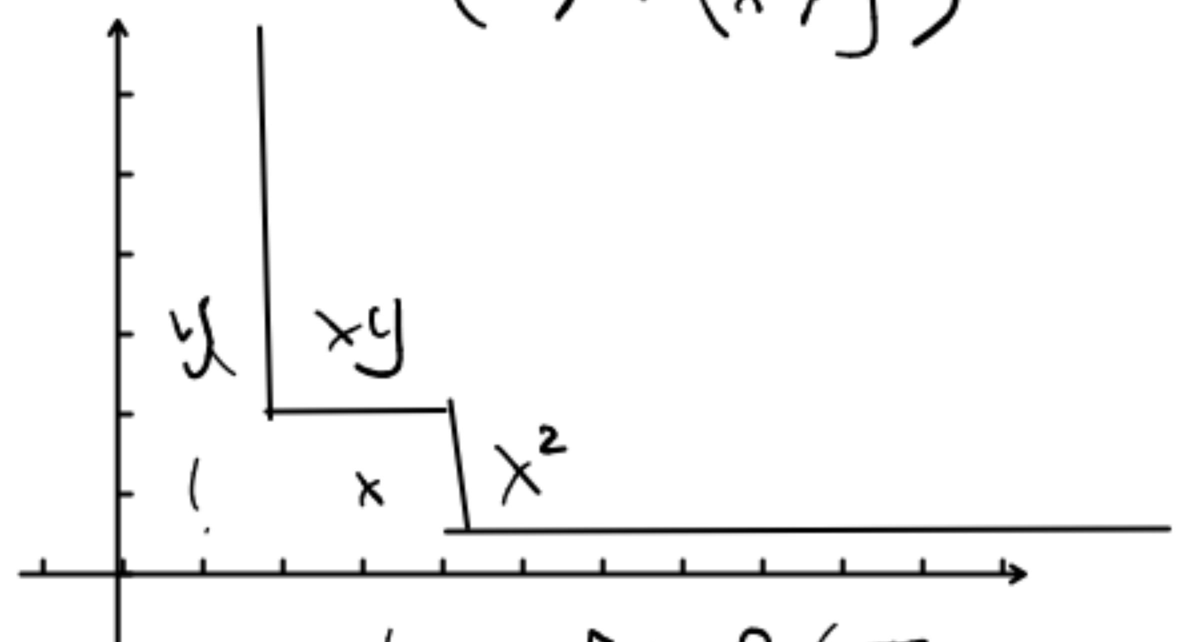
Definition. A <u>monomial</u> of S is a product $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = x^a$ where $a = (a_1, \dots, a_n) = \partial(x^a)$ is the <u>multidegree</u>.

Perhaps it is sometimes a scalar multiple?
A scalar multiple is called a <u>term</u>.

A monomial ideal I is one generated by monomials.
It has a k-basis of monomials.

We have seen e.g. we can compute intersections of m. ideals.

$$(x^2, xy) = (x) \cap (x^2, xy, y^2)$$
$$= (x) \cap (x^2, y)$$



We see a basis for S/I
e.g. $\bar{1}, \bar{x}, \bar{y}, \bar{y}^2, \dots$

We can easily compute the Hilbert function & Poincaré series of S/I.

We see that monomial ideals are finitely generated.

In fact we know ideals of S are all finitely gen'd.

Gordan's 1900 proof of Hilbert's basis theorem used this.

Proof of Hilbert's basis theorem

Definition. A basis for an ideal is a set of ideal generators ~~for~~ the ideal.

Hilbert's Basis Theorem.
If R is a Noetherian ring then so is the polynomial ring R[x].
Every ideal of R[x] has a finite basis.

Proof.
Let $I \subset R[x]$ be an ideal.
Let
$L = \{$leading coefficients of elements of $I\}$
Claim: this is an ideal of R.
(Proof $f = ax^d + \text{lower} \in I$
$g = bx^e + \text{lower} \in I$
then $ra - b$ is either 0 or the leading coeff of $rx^e f - x^d g$)
L is finitely generated by
$a_1, \ldots, a_n \in R$.
Let $f_i \in I$ have leading coeff a_i
Put $e_i = \deg f_i$, $N = \max\{e_1, \ldots, e_n\}$
If $0 \le d \le N-1$ put
$L_d = \{$leading coeffs of polys in I of degree $d\}$
This is also an ideal.
$L_d = (b_{d,1}, \ldots, b_{d,n_d})$. $b_{ij} \in R$
Find $f_{d,i} \in I$ of degree d
with leading coeff $b_{d,i}$

Claim :

$I = (\{f_1, \dots, f_n\} \cup \{f_{d,i} \mid 0 \le d < N, 1 \le i \le n_d\})$

(Pf. Let I' be the ideal on the right.

$I' \subseteq I$. If $\neq$, pick $f \in I - I'$ of least degree.

If $\deg f \geq N$ then its leading coeff is a combn of $a_1, \dots, a_n$.

Let g = same combn of $x^{\deg f - \deg f_i} f_i \in I$

Now $f - g \in I - I'$ has smaller degree than f.

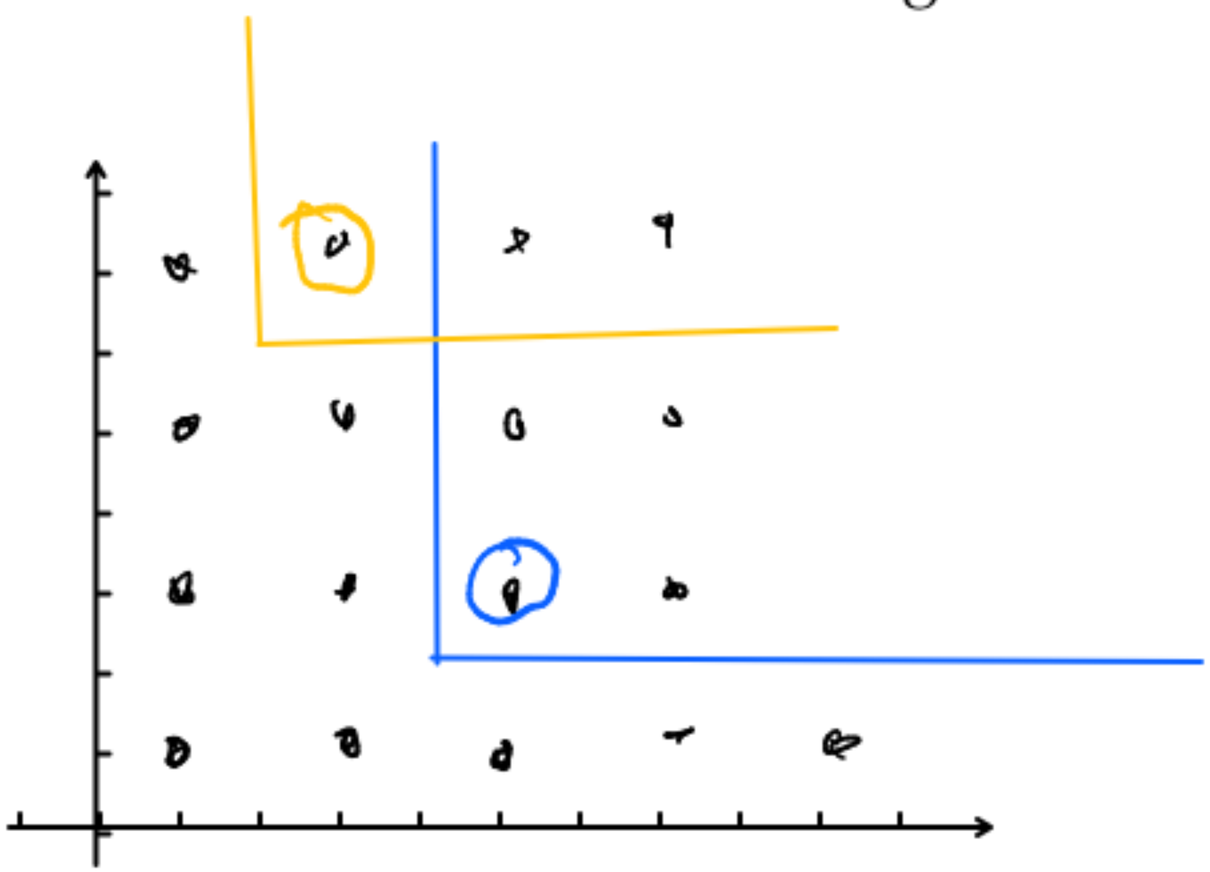
Contradiction.

Similar if $\deg f < N$. Find g = combn of $f_{\deg f, i} \in I'$ with same leading term as f. Now $f - g \in I - I'$ has smaller degree than f. contradiction. $\square$

Pre-class Warm-up!!

Consider the following:

Proposition.
Monomial ideals of $S = k[x_1, \dots ,x_d]$ satisfy ACC:
If J is a monomial ideal of S, every set of monomials that generates it contains a finite set of monomials that generates it.



Use Dickson's Lemma:
Given infinitely many vectors $v_1, v_2, \dots$ in N^r, there exists $i < j$ with $v_i \leq v_j$, where $\leq$ means coordinate-by-coordinate comparison.

This means $\mathbb{N}^r$

Any sequence in $\mathbb{N}$ contains a weakly increasing sequence.

Monomial orders

Recall: $S = k[x_1, \ldots, x_n]$.
A monomial is an expression $x^a = x_1^{a_1} \cdots x_n^{a_n}$
A term is a scalar multiple of a monomial.

Definition.
A monomial ordering is one of the following equivalent relations on the set of monomials:

1. A well-ordering $\geq$ on {monomials} such that $u \geq v$ implies $mu \geq mv$ always.

2. A total order on {monomials} such that $u \geq v$ implies $mu \geq mv$, and $m \geq 1$ always.

Is it obvious that 1. $\Rightarrow m \geq 1 \ \forall m$?

Examples of monomial orders:

The lexicographic order:
$x_1 > x_2 > \cdots > x_n$
$x_1^{a_1} \cdots x_n^{a_n} \geq x_1^{b_1} \cdots x_n^{b_n}$
$\Leftrightarrow$ the earliest $a_i \neq b_i$ has $a_i \geq b_i$

Homogeneous lexicographic
$u > v \Leftrightarrow \deg u > \deg v$
or $\deg u = \deg v$ and $u >_{lex} v$.

More definitions. Let $f \in S$

Fix a monomial ordering on $S = k[x_1, \dots, x_n]$.

Extend the order to terms.

The leading term (or initial term) $LT(f)$ is

the largest term in f.

If I is an ideal of S, the ideal of leading terms is

$LT(I) = (LT(f) \mid f \text{ is in } I)$

= the ideal generated by the leading terms of polynomials in I.

It is a monomial ideal.

Is it obvious that every monomial in $LT(I)$ is the LT of some $f \in I$?

Examples (page 318 of D & F)

$S = k[x,y]$. Lexicographic order $x > y$.

Let $f = x^3y - xy^2 + 1$, $g = x^2y^2 - y^3 - 1$

	f	g
LT	x^3y	x^2y^2
∂	$(3,1)$	$(2,2)$

Observe $\underbrace{yf - xg}_{LT = x} = x + y$ lies in $J = (f,g)$

We see: $LT(J) \neq (LT(f), LT(g))$. $\not\ni x$

Question: if $y > x$, what are $LT(f)$ and $LT(g)$?

$LT(f) = -xy^2$ $\quad$ $LT(g) = -y^3$

Proposition (Macaulay, see 15.3 of Eisenbud)
Let J be an ideal of S.
The (images of the) monomials of S not in $LT(J)$ are a k-linear basis for S / J.

Proof. Let B be the set of monomials not in $LT(J)$

They are lin. ind modulo J

If $p = \sum_{m_i \in B} u_i m_i \in J$, $u_i \in k$.

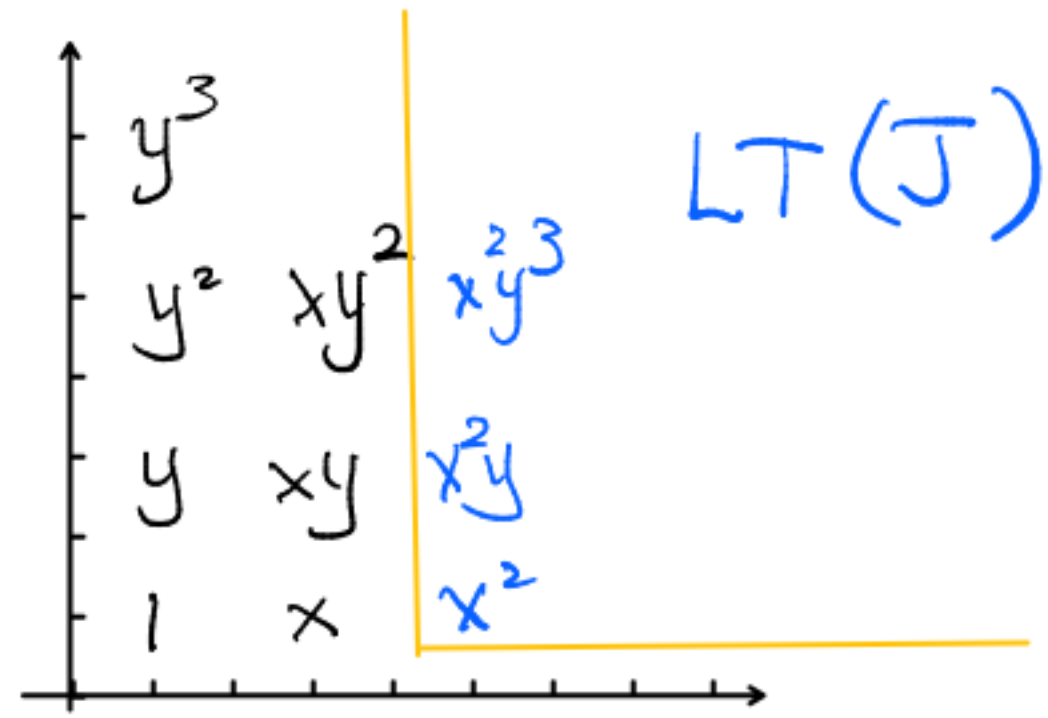
$LT(p) \in LT(J)$. $LT(p)$ is one of the $m_i \notin LT(J)$. Contradiction

We show They span: $\underbrace{\langle B \rangle}_{\text{span of } B} + J = S$

If $\neq S$, pick $f \in S - (\langle B \rangle + J)$ with $LT(f)$ minimal. If $LT(f) \in B$ Then $f - LT(f) \in S - (\langle B \rangle + J)$ has smaller LT. O/w $LT(f) = LT(g)$ $g \in J$. Now $f - g \notin \langle B \rangle + J$, and has smaller LT.

Example: $J = (x^2 - y^3)$
$x > y$
(x^2)

The monomials not in (x^2) do give a basis for S/J.



Each f in S determines one of these basis elements as the coset representative of its coset $f + J$. Groebner methods give a way to compute this, and in particular determine whether f is in J.

We can compute Samuel functions.

Definition.
A Groebner basis for an ideal J in S is a finite set g_1, … , g_d of elements of J so that the leading terms LT(g_1), … , LT(g_d) generate LT(J).

Examples.

1. $J = (x^2 - y^3)$ has $x^2 - y^3$ as G. basis because $LT(J) = (LT(x^2 - y^3))$

2. $J = (f, g)$ as before doesn't have f, g as a G. basis In fact $x+y$, and another polynomial in J with $LT = y^4$ is a G. basis.

Proposition. If g_1, … , g_d is a Groebner basis, it generates J.

Proof.
Let g_1, … ,g_d be a Groebner basis for J and let L = (g_1, … ,g_d) be the ideal it generates, so L is contained in J.
Pick f in J - L with least leading term among such f. Write LT(f) = LT(g) for some polynomial g in L. Then f - g lies in J - L has smaller LT, a contradiction.

Note: $LT(L) = LT(J)$ because it is generated by LTs of polynomials in L.

Corollary.

Proof of Hilbert's basis theorem for S

Theorem.
When k is a field, every ideal of $S = k[x_1, \dots, x_d]$ is finitely generated.

Proof. Let J be an ideal of S.

General polynomial division

Fix a monomial ordering on S.
Let $g_1, \ldots, g_m$ be a set of non-zero polynomials.
Let f be a polynomial in S.
We will work with 'quotients' q_i and a 'remainder' r so that at the end

$$f = q_1 g_1 + \cdots + q_m g_m + r$$

Each $q_i g_i$ has multi degree $\leq \partial(f)$.
The remainder r has no nonzero term divisible by any $LT(g_i)$.

Start with the q_i and r all equal to 0.
Successively test whether the leading term of the dividend f is divisible by the leading terms of the divisors $g_1, \ldots, g_m$, in that order.

Step 1. If $LT(f)$ is divisible by $LT(g_i)$, say, $LT(f) = a_i LT(g_i)$, ad a_i to the quotient q_i, replace f by the dividend $f - a_i g_i$ (a polynomial with lower order LT) and reiterate the entire process.

Step 2. If the leading term of the dividend f is not divisible by any of the leading terms $LT(g_1)$, n… , $LT(g_m)$, add the leading term of f to the remainder r, replace f by the dividend $f - LT(f)$, and reiterate the entire process