Math 8300 **Homework 1** **due Thursday 9/30/2021**

1 Let V be the 2-dimensional representation of the symmetric group S_3 over $\mathbb{F}_2$ where the permutations $(1,2)$ and $(1,2,3)$ act via matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Show that V is simple.

2. (The modular law.) Let A be a ring and $U = V \oplus W$ an A-module which is the direct sum of A-modules V and W. Show by example that if X is any submodule of U then it need not be the case that $X = (V \cap X) \oplus (W \cap X)$. Show that if we make the assumption that $V \subseteq X$ then it is true that $X = (V \cap X) \oplus (W \cap X)$.

3. Let V be the 3-dimensional permutation representation of the symmetric group S_3 over $\mathbb{F}_3$, where the permutations $(1,2)$ and $(1,2,3)$ act via matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that V has a unique subrepresentation of dimension 1, and a unique subrepresentation of dimension 2.

4. Let V be an A-module for some ring A and suppose that V is a sum $V = V_1 + \cdots + V_n$ of simple submodules. Assume further that the V_i are pairwise non-isomorphic. Show that the V_i are the only simple submodules of V and that $V = V_1 \oplus \cdots \oplus V_n$ is their direct sum.

5. Let

$$\rho_1 : G \to GL(V)$$
$$\rho_2 : G \to GL(V)$$

be two representations of G on the same vector space V which are injective as homomorphisms. (One says that such a representation is *faithful*.) Consider the three statements

(a) the RG-modules given by ρ_1 and ρ_2 are isomorphic,
(b) the subgroups $\rho_1(G)$ and $\rho_2(G)$ are conjugate in $GL(V)$,
(c) for some automorphism $\alpha \in \text{Aut}(G)$ the representations ρ_1 and $\rho_2\alpha$ are isomorphic.

Show that (a) $\Rightarrow$ (b) and that (b) $\Rightarrow$ (c).

6. One form of the Jordan-Zassenhaus theorem states that for each n, $GL(n,\mathbb{Z})$ (that is, $\text{Aut}(\mathbb{Z}^n)$) has only finitely many conjugacy classes of subgroups of finite order. Assuming this, show that for each finite group G and each integer n there are only finitely many isomorphism classes of representations of G on $\mathbb{Z}^n$.

7. Let $\phi : U \to V$ be a homomorphism of A-modules, where A is a ring. Show that $\phi(\operatorname{soc} U) \subseteq \operatorname{soc} V$. Show that ϕ is one-to-one if and only if the restriction of ϕ to $\operatorname{soc} U$ is one-to-one. Show that if ϕ is an isomorphism then ϕ restricts to an isomorphism $\operatorname{soc} U \to \operatorname{soc} V$.

Extra questions: Do **not** hand in.

8. Let $G = C_p = \langle x \rangle$ be cyclic of prime order p and $R = \mathbb{F}_p$ for some prime p. Consider the two representations ρ_1 and ρ_2 specified by

$$\rho_1(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_2(x) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate the socles of these two representations. Show that the second representation is the direct sum of two non-zero subrepresentations.

9. Let k be an infinite field of characteristic 2, and $G = \langle x, y \rangle \cong C_2 \times C_2$ be the non-cyclic group of order 4. For each $\lambda \in k$ let $\rho_\lambda(x), \rho_\lambda(y)$ be the matrices

$$\rho_\lambda(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \rho_\lambda(y) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

regarded as linear maps $U_\lambda \to U_\lambda$ where U_λ is a k-vector space of dimension 2 with basis $\{e_1, e_2\}$.

(a) Show that ρ_λ defines a representation of G with representation space U_λ.

(b) Find a basis for $\operatorname{soc} U_\lambda$.

(c) By considering the effect on $\operatorname{soc} U_\lambda$, show that any kG-module homomorphism $\alpha : U_\lambda \to U_\mu$ has a triangular matrix $\alpha = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ with respect to the given bases.

(d) Show that if $U_\lambda \cong U_\mu$ as kG-modules then $\lambda = \mu$. Deduce that kG has infinitely many non-isomorphic 2-dimensional representations.