Math 8300 **Solutions 1** **Thursday 9/30/2021**

1 Let V be the 2-dimensional representation of the symmetric group S_3 over $\mathbb{F}_2$ where the permutations $(1,2)$ and $(1,2,3)$ act via matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Show that V is simple.

Solution: We show that V is generated as an $\mathbb{F}_2 S_3$-module by every non-zero vector it contains. There are 3 such vectors, namely the transposes of $(1,0),(0,1)$ and $(1,1)$. Applying $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to the first two gives the other vector in the standard basis of V. Applying $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ to the transpose of $(1,1)$ gives the transpose of $(1,0)$, which is already shown to generate V, so all three vectors generate V. Thus V is simple.

2. (The modular law.) Let A be a ring and $U = V \oplus W$ an A-module which is the direct sum of A-modules V and W. Show by example that if X is any submodule of U then it need not be the case that $X = (V \cap X) \oplus (W \cap X)$. Show that if we make the assumption that $V \subseteq X$ then it is true that $X = (V \cap X) \oplus (W \cap X)$.

Solution: Let k be a field and let $U = k^2$, V the span of the transpose of $(1,0)$, W the span of the transpose of $(0,1)$, and X the span of the transpose of $(1,1)$. Then $V \cap X = W \cap X = 0$, so that $X \neq (V \cap X) \oplus (W \cap X)$, but $U = V \oplus W$.
If we suppose that $V \subseteq X$, to show that $X = (V \cap X) \oplus (W \cap X)$, note first that $(V \cap X) \cap (W \cap X) \subseteq V \cap W = 0$. We show that $X = (V \cap X) + (W \cap X)$. We can write any vector $u \in X$ uniquely as $u = v + w$ where $v \in V$ and $w \in W$. Because $V \subseteq X$ we see that $v \in X$, so $w = u - v \in X$ so $u \in (V \cap X) + (W \cap X)$.

3. Let V be the 3-dimensional permutation representation of the symmetric group S_3 over $\mathbb{F}_3$, where the permutations $(1,2)$ and $(1,2,3)$ act via matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Show that V has a unique subrepresentation of dimension 1, and a unique subrepresentation of dimension 2.

Solution: We start by observing that there are invariant subspaces

$$U = \mathbb{F}_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \subset \{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a + b + c = 0\} = W$$

of dimensions 1 and 2, and we show that these are the only proper invariant subspaces. Writing vectors as row vectors, suppose a vector (a, b, c) spans a 1-dimensional subspace. Applying the powers of the 3-cycle $(1, 2, 3)$ there is a scalar λ so that $(b, c, a) = \lambda(a, b, c)$, and $\lambda^3 = 1$, which implies $\lambda = 1$. Thus $a = b = c$ and we deduce that the only invariant 1-dimensional subspace is U, spanned by $(1, 1, 1)$. Now suppose there is a 2-dimensional invariant subspace $W_1 \neq W$. Then $W_1 \cap W$ has dimension 1 by the rank formula, and is invariant, so $W_1 \cap W = U$. If W_1 contains a vector (a, b, c) not in W, then also $(a, b, c) - c(1, 1, 1) = (a - c, b - c, 0)$ is not in W, as is $(a - c, b - c, 0) - (a - c)(1, -1, 0)$, which is a non-zero scalar multiple of $(1, 0, 0)$. Thus W_1 contains the three standard basis vectors of V, so equals V (contradicting the dimension of W_1 being 2). This shows that U and W are the only proper invariant subspaces.

4. Let V be an A-module for some ring A and suppose that V is a sum $V = V_1 + \cdots + V_n$ of simple submodules. Assume further that the V_i are pairwise non-isomorphic. Show that the V_i are the only simple submodules of V and that $V = V_1 \oplus \cdots \oplus V_n$ is their direct sum.

   Solution. We know from class that V is a direct sum of a subset of the V_i. From the direct sum we can construct a composition series with this subset of the V_i as composition factors. Each of the V_i does appear in some composition series, because every submodule is part of a composition series, so each V_i is a composition factor. By the Jordan-Hölder theorem, all the V_i must appear in the direct sum, so V is the direct sum of all of them. Let S be a simple submodule of V. Then S is also a composition factor of V and must be isomorphic to some V_i. We have either $V_i \cap S = 0$ or $V_i \cap S = V_i = S$ because both V_i and S are simple. In the first case, the submodule of V generated by V_i and S is $V_i \oplus S$, so that V_i appears as a composition factor of V with multiplicity 2, which does not happen. Thus $S = V_i$, and the only simple submodules of V are the V_i.

5. Let
$$\rho_1 : G \to GL(V)$$
$$\rho_2 : G \to GL(V)$$
   be two representations of G on the same vector space V which are injective as homomorphisms. (One says that such a representation is *faithful*.) Consider the three statements
   (a) the RG-modules given by ρ_1 and ρ_2 are isomorphic,
   (b) the subgroups $\rho_1(G)$ and $\rho_2(G)$ are conjugate in $GL(V)$,
   (c) for some automorphism $\alpha \in \mathrm{Aut}(G)$ the representations ρ_1 and $\rho_2\alpha$ are isomorphic.
   Show that (a) $\Rightarrow$ (b) and that (b) $\Rightarrow$ (c).

   Solution: Suppose (a) holds. Then there is an invertible linear map $\theta : V \to V$ so that, for all $v \in V$ and $g \in G$, $\rho_2(g)(\theta(v)) = \theta\rho_1(g)(v)$. Thus $\theta\rho_1(g)\theta^{-1}(w) = \rho_2(g)(w)$ for all $winV$ and $g \in G$, which means that $\theta\rho_1(g)\theta^{-1} = \rho_2(g)$ for all $g \in G$. This implies that the subgroups $\rho_1(G)$ and $\rho_2(G)$ are conjugate.
   Suppose (b) holds. Then, for some $\theta \in GL(V)$ we have $\rho_2(G) = \theta\rho_1(G)\theta^{-1}$, which we can write as $\rho_2(G) = c_\theta\rho_1(G)$ where $c_\theta : GL(V) \to GL(V)$ is the map $c_\theta(\beta) = \theta\beta\theta^{-1}$. Thus ρ_2

and $c_\theta\rho_1$ have the same image, but might not be the same map, and they are one-to-one. Now $\alpha := \rho_2^{-1} c_\theta \rho_1 \in \mathrm{Aut}(G)$ and $\rho_2\alpha = c_\theta\rho_1$, where ρ_2^{-1} means the inverse of ρ_2 on its image. By the same calculations as in the first implication, this means θ is an isomorphism between ρ_1 and $\rho_2\alpha$.

6. One form of the Jordan-Zassenhaus theorem states that for each n, $GL(n,\mathbb{Z})$ (that is, $\mathrm{Aut}(\mathbb{Z}^n)$) has only finitely many conjugacy classes of subgroups of finite order. Assuming this, show that for each finite group G and each integer n there are only finitely many isomorphism classes of representations of G on $\mathbb{Z}^n$.

   Solution. We retain the notation c_θ from queston 5. The Jordan-Zassenhaus theorem implies that, for each finite group G, there are only finitely many equivalence classes of homomorphisms $G \to GL(n,\mathbb{Z})$ under the relation $\rho_1 \sim \rho_2$ if and only if $\rho_2(G) = c_\theta\rho_1(G)$ for some $\theta \in GL(n,\mathbb{Z})$. Because there are only finitely many maps between two finite sets, it follows that there are only finitely many equivalence classes of homomorphisms $G \to GL(n,\mathbb{Z})$ under the relation $\rho_1 \sim' \rho_2$ if and only if $\rho_2 = c_\theta\rho_1$ for some $\theta \in GL(n,\mathbb{Z})$. Such equivalence classes biject with isomorphism classes of representations of G on $\mathbb{Z}^n$, by the same argument as in question 5.

7. Let $\phi : U \to V$ be a homomorphism of A-modules, where A is a ring. Show that $\phi(\mathrm{Soc}\, U) \subseteq \mathrm{Soc}\, V$. Show that ϕ is one-to-one if and only if the restriction of ϕ to $\mathrm{Soc}\, U$ is one-to-one. Show that if ϕ is an isomorphism then ϕ restricts to an isomorphism $\mathrm{Soc}\, U \to \mathrm{Soc}\, V$.

   Solution: We can write $\mathrm{Soc}\, U = \sum_i S_i$ where the S_i are simple submodules of U. Now $\phi(\mathrm{Soc}\, U) = \phi(\sum_i S_i) = \sum_i \phi(S_i)$ and this is a sum of simple modules because each $\phi(S_i)$ is either simple or zero. It follows that $\phi(\mathrm{Soc}\, U) \subseteq \mathrm{Soc}\, V$, the largest sum of simple submodules of V.
   If ϕ is one-to-one then its restriction to any subset is one-to-one. Conversely, if ϕ is not one-to-one then $\mathrm{Ker}\, \phi$ has a simple submodule, which is contained in $\mathrm{Soc}\, U$, so ϕ is not one-to-one on restriction to $\mathrm{Soc}\, U$.
   If ϕ is an isomorphism then $\phi^{-1} : \mathrm{Soc}\, V \to \mathrm{Soc}\, U$ so that ϕ and ϕ^{-1} restrict to inverse isomorphisms between the socles.