Math 8300 **Homework 2** **due Thursday 10/28/2021**
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1. Let A be a ring with a 1, and let V be an A-module. An element e in any ring is called *idempotent* if and only if $e^2 = e$.
(a) Show that an endomorphism $e : V \to V$ is a projection onto a subspace W if and only if e is idempotent as an element of $\mathrm{End}_A(V)$. (The term *projection* means a linear mapping onto a subspace that is the identity on restriction to that subspace.)
(b) Show that direct sum decompositions $V = W_1 \oplus W_2$ as A-modules are in bijection with expressions $1 = e + f$ in $\mathrm{End}_A(V)$, where e and f are idempotent elements with $ef = fe = 0$. (In case $ef = fe = 0$, e and f are called *orthogonal*.)

Solution. (a) If e is a projection then, for any vector v we have $e(e(v)) = e(v)$ because e is the identity on its image, so that $e^2 = e$. Conversely, if $e^2 = e$ then for any vector v we have $e(e(v)) = e(v)$ so e is the identity on its image.
(b) Given a direct sum decomposition $V = W_1 \oplus W_2$ let e be the endomorphism of V that is $e(w_1, w_2) = w_1$ and let $f(w_1, w_2) = w_2$. Then $1 = e + f$, $e^2 = e$, $f^2 = f$, $ef = fe = 0$. Conversely, given such e, f we define $W_1 = e(V)$, $W_2 = f(V)$. Then $V = W_1 + W_2$ because if $v \in V$ then $v = 1(v) = e(v) + f(v) \in W_1 + W_2$. Also if $v \in W_1 \cap W_2$ then $v = e(x) = f(y)$ for vectors x, y. Then $v = e(x) = e(e(x)) = e(f(y)) = 0$. Thus $V = W_1 \oplus W_2$. These two constructions are mutually inverse.

2. Consider a ring with identity that is the direct sum (as a ring) of non-zero subrings $A = A_1 \oplus \cdots \oplus A_r$.
(a) Writing $1_A = u_1 + \cdots + u_r$ with $u_i \in A_i$, show that the elements u_i are idempotent.
(b) Suppose that A has exactly n isomorphism types of simple modules. Show that $r \le n$.

Solution. (a) If $i \ne j$ then $u_i u_j = 0$ because these elements lie in different summands, and multiplication is summand by summand. Then $u_i = 1_A u_i = (u_1 + \cdots + u_r)u_i = u_i u_i$ because the other products are zero.
(b) Each A_i has at least one simple module S_i (and possibly more) constructed as $S_i = A_i/J_i$ where J_i is a maximal left ideal of A_i. Each S_i becomes an A-module by letting the factors A_j with $j \ne i$ act as 0, and it is simple as an A-module. If $i \ne j$ then S_i and S_j are non-isomorphic because u_i acts as the identity on one of them and as 0 on the other. Thus A has at least r simple modules.

3. Let g be any non-identity element of a group G. Show that G has a simple complex character χ for which $\chi(g)$ has negative real part.

Solution. The real part of the dot product of the columns indexed by 1 and g of the character table is a positive integral combination of the real parts of the $\chi(g)$, and it must be 0. Because the trivial character has value 1 at g it means that some other real part must be negative.

4. Suppose that V is a representation of G over $\mathbb{C}$ for which $\chi_V(g) = 0$ if $g \neq 1$. Show that $\dim V$ is a multiple of $|G|$. Deduce that $V \cong \mathbb{C}G^n$ for some n. Show that if W is any representation of G over $\mathbb{C}$ then $\mathbb{C}G \otimes_{\mathbb{C}} W \cong \mathbb{C}G^{\dim W}$ as $\mathbb{C}G$-modules.

Solution. The character $\psi = \sum d_i \chi_i$ of the regular representation (where d_i is the degree of χ_i) has values $\psi(1) = |G|$ and $\psi(g) = 0$ if $g \neq 1$. From independence of the characters it follows that $\chi_V = \sum \frac{\dim V}{|G|} d_i \chi_i$ is the unique expression for χ_V as a linear combination of the simple characters. Because $d_1 = 1$ we deduce that $\frac{\dim V}{|G|}$ is an integer, as required. Now χ_V is the character of $\mathbb{C}G^n$ where $n = \frac{\dim V}{|G|}$, so $V \cong \mathbb{C}G^n$ because characters determine representations. Finally, if W is any (finite dimensional, but actually it doesn't matter) representation of G then the character of $\mathbb{C}G \otimes_{\mathbb{C}} W$ at g is $\chi_{\mathbb{C}G}(g)\chi_W(g) = 0\chi_W(g) = 0$ if $g \neq 1$ and the dimension of $\mathbb{C}G \otimes_{\mathbb{C}} W$ is $|G| \dim W$, so $\mathbb{C}G \otimes_{\mathbb{C}} W \cong \mathbb{C}G^{\dim W}$.

5. Show that if every element of a finite group G is conjugate to its inverse, then every character on G is real-valued.
Conversely, show that if every character on G is real-valued, then every element of G is conjugate to its inverse.
[Extra irrelevant information: it is possible to have a group G in which every element is conjugate to its inverse, but not every complex representation of G is equivalent to a real representation.]

Solution. If g and g^{-1} lie in the same conjugacy class then for every character, $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$ so $\chi(g)$ is real. Conversely, if every character on G is real-valued, because the columns of the character table corresponding to any element g and its inverse g^{-1} are complex conjugates of each other, they are the same. Also, the columns of the character table are linearly independent, so $g = g^{-1}$.

6. Let G permute a set Ω and let $R\Omega$ denote the permutation representation of G over R determined by Ω. This means $R\Omega$ has a basis in bijection with Ω and each element $g \in G$ acts on $R\Omega$ by permuting the basis elements in the same way that g permutes Ω.
(a) Show that when H is a subgroup of G and $\Omega = G/H$ is the set of left cosets of H in G, the kernel of G in its action on $R\Omega$ is H if and only if H is normal in G.
(b) Show that the normal subgroups of G are precisely the subgroups of the form $\operatorname{Ker} \chi_{i_1} \cap \cdots \cap \operatorname{Ker} \chi_{i_t}$ where $\chi_1, \ldots, \chi_n$ are the simple characters of G. Deduce that the normal subgroups of G are determined by the character table of G.
(c) Show that G is a simple group if and only if for every non-trivial simple character χ and for every non-identity element $g \in G$ we have $\chi(g) \neq \chi(1)$.

Solution. (a) An element $u \in G$ is in the kernel of the action of $R\Omega$ if and only if it fixes each coset gH under left multiplication: $ugH = gH$. This means $g^{-1}ugH = H$, or $g^{-1}ug \in H$, so $u \in gHg^{-1}$. The kernel is H if and only if $H \subseteq gHg^{-1}$ for all $g \in G$, if and only if H is normal in G.
(b) The intersections of kernels are always normal subgroups. Conversely, if H is a normal subgroup then H is the kernel of the action on $R[G/H]$, which is the common kernel, or

intersection of the kernels, of the simple constituents of $R[G/H]$. Thus all normal subgroups arise as intersections of kernels of simple representations. The kernel of a (simple) character χ is determined as the set of elements g for which $\chi(g) = \chi(1)$, so the normal subgroups are determined by the character table.
(c) By part (b), G is simple if and only if the kernel of every simple character is either G or 1. The only character with kernel G is the trivial character, so G is simple if and only if all non-trivial characters have kernel 1. The non-trivial characters have kernel 1 if and only if there is no non-identity element $g \in G$ with $\chi(1) = \chi(g)$, and this finishes the proof.

7. While walking down the street you find a scrap of paper with the following character table on it:

$$\begin{array}{ccccc}
\hline
 & 1 & & 1 & \\
 & 1 & & -1 & \\
\cdots & 2 & \cdots & -1 & \cdots \\
 & 3 & & 1 & \\
 & 3 & & -1 & \\
\end{array}$$

All except two of the columns are obscured, and while it is clear that there are five rows you cannot read anything of the other columns, including their position. Prove that there is an error in the table. Given that there is exactly *one* error, determine where it is, and what the correct entry should be.

Solution. The dot product of the two columns is -2, so there must be an error. Suppose the error is in the first of the visible columns. It is not in the first place, because there would be no trivial character. Dot product with the other column shows that the correct entries would have to be -1 in row 2, or 0 in row 3 (not possible, because this column is no longer the column of degrees, and dot product with the column of degrees would not be 0), or 5 in row 4 (not possible because this must be the column of degrees, and tensoring with the non-trivial degree 1 character does not preserve simple characters), or 1 in row 5 (not possible because this must be the column of degrees, and tensoring with the non-trivial degree 1 character does not preserve simple characters). All possibilities are eliminated except -1 in row 2. In this case the first visible column is not the column of degrees. Writing the character degrees as $[a, b, c, d, e]$, dot product with the two visible columns give $a - b + 2c + 3d + 3e = 0$ and $a - b - c + d - e = 0$ from which we deduce $3c + 2d + 4e = 0$, which is not possible because c, d, e are positive integers.
We conclude there is no error in the first visible column. We deduce that that column must be the column of degrees. Tensoring with the non-trivial degree 1 character preserves simple characters, sending the degree 2 character to itself, so we deduce that the error is in row 3 of the second column, where the value must be 0.

8. A finite group has seven conjugacy classes of elements with representatives $c_1, \ldots, c_7$ (where $c_1 = 1$), and the values of five of its irreducible characters are given by the following

table:

c_1	c_2	c_3	c_4	c_5	c_6	c_7
1	1	1	1	1	1	1
1	1	1	1	-1	-1	-1
4	1	-1	0	2	-1	0
4	1	-1	0	-2	1	0
5	-1	0	1	1	1	-1

Calculate the numbers of elements in the various conjugacy classes and the remaining simple characters.

Solution. The Kronecker product of the non-identity degree 1 character with any simple character is a simple character, and doing this with the degree 5 character gives another character with values $[5, -1, 0, 1, -1, -1, 1]$. There remains one character $[a, b, c, d, e, f, g]$, say, where a is a positive integer. Taking the dot product of columns c_1 and c_3 gives $ac = 6$, from which we deduce the degree of the last character is $a = 1, 2, 3$ or 6. Computing the sum of the squares of the degrees, we get $|G| = 85, 88, 93$ or 120. The only one of these divisible by the degrees 4 and 5 is 120, so we deduce the last character has degree 6 and $|G| = 120$. Column orthogonality now gives the last character as $[6, 0, 1, -2, 0, 0, 0]$. Dot products of columns with themselves gives the centralizer orders, whose indices are the conjugacy class sizes: $[1, 20, 24, 15, 10, 20, 30]$.

Extra questions: do not upload to Gradescope

9. Let $g \in G$.

(a) Prove that g lies in the center of G if and only if $|\chi(g)| = |\chi(1)|$ for every simple complex character χ of G.

(b) Show that if G has a faithful simple complex character (one whose kernel is 1) then the center of G is cyclic. (You may assume that every finite subgroup of $\mathbb{C}$ is cyclic.)

10. Let U be a module for a semisimple finite dimensional algebra A. Show that if $\mathrm{End}_A(U)$ is a division ring then U is simple.

11.(a) By using characters show that if V and W are $\mathbb{C}G$-modules then $(V \otimes_{\mathbb{C}} W)^* \cong V^* \otimes_{\mathbb{C}} W^*$, and $({}_{\mathbb{C}G}\mathbb{C}G)^* \cong {}_{\mathbb{C}G}\mathbb{C}G$ as $\mathbb{C}G$-modules.

(b) If k is any field and V, W are kG-modules, show that $(V \otimes_k W)^* \cong V^* \otimes_k W^*$, and $({}_{kG}kG)^* \cong {}_{kG}kG$ as kG-modules. (Can you guess maps that might be isomorphisms?)

12. Here is a column of a character table:

$$\begin{array}{c} g \\ \hline 1 \\ -1 \\ 0 \\ -1 \\ -1 \\ \frac{-1+i\sqrt{11}}{2} \\ \frac{-1-i\sqrt{11}}{2} \\ 0 \\ 1 \\ 0 \end{array}$$

(a) Find the order of g.
(b) Prove that $g \notin Z(G)$.
(c) Show that there exists an element $h \in G$ with the same order as g but not conjugate to g.
(d) Show that there exist two distinct simple characters of G of the same degree.