Math 8300 **Homework 3** **due Saturday 11/27/2021**
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1. Let G be the non-abelian group of order 21:

$$G = \langle x, y \mid x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle.$$

Show that G has 5 conjugacy classes, and find its character table. [The answer and a brief hint are given in the character tables section at the end of my text book. If you look there, make sure to present the calculations that are suggested.]
Solution. For the purposes of this question we can accept that G has order 21. The last relation shows that the subgroup $\langle x \rangle$ is normal, with three conjugacy classes $\{1\}$, $\{x, x^2, x^4\}$ (because $x^8 = x$) and $\{x^3, x^5, x^7\}$. The other 14 elements lie in the two cosets $\langle x \rangle y$ and $\langle x \rangle y^{-1}$. From the relation $yxy^{-1} = x^2$ we get $yx = x^2 y$ and so $x^{-1}yx = x^{-1}x^2 y = xy$, and inductively $x^r y x^{-r} = x^r y$. These elements exhaust the coset $\langle x \rangle y$ and similarly the conjugates of y^{-1} form the coset $\langle x \rangle y^{-1}$ (bearing in mind that $\langle x \rangle$ is normal. This gives the 5 conjugacy classes in the following table (where $\zeta_n = e^{2\pi i/n}$):

$C_7 \rtimes C_3$
ordinary characters

g $\lvert C_G(g)\rvert$	1 21	x 7	x^{-1} 7	y 3	y^{-1} 3
χ_1	1	1	1	1	1
χ_{1a}	1	1	1	ζ_3	ζ_3^2
χ_{1b}	1	1	1	ζ_3^2	ζ_3
χ_{3a}	3	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	$\zeta_7^3 + \zeta_7^5 + \zeta_7^6$	0	0
χ_{3b}	3	$\zeta_7^3 + \zeta_7^5 + \zeta_7^6$	$\zeta_7 + \zeta_7^2 + \zeta_7^4$	0	0

The three 1-dimensional characters are representations of the abelianization $C_3 = G/\langle x \rangle$. For the remaining two characters we take the characters χ_{ζ_7} and $\chi_{\zeta_7^3}$ of $\langle x \rangle$ and induce them to G, The formula for an induced character $\psi \uparrow_{\langle x \rangle}^G$ is the sum of the values $\psi(g) + \psi({}^y g) + \psi({}^{y^{-1}} g)$ when these group elements lie in $\langle x \rangle$, with 0 as the value when they don't, and this accounts for the two 3-dimensional characters. To see they are simple we may compute

$$\langle \psi \uparrow_{\langle x \rangle}^G, \psi \uparrow_{\langle x \rangle}^G \rangle_G = \langle \psi, \psi \uparrow_{\langle x \rangle}^G \downarrow_{\langle x \rangle}^G \rangle_{\langle x \rangle} = \langle \psi, \psi + {}^y\psi + {}^{y^2}\psi \rangle_{\langle x \rangle}$$

using Frobenius reciprocity and Mackey's formula, and the answer is 1 if $\psi = \chi_{\zeta_7}$ or $\chi_{\zeta_7^3}$ because then the characters ψ, ${}^y\psi$ and ${}^{y^2}\psi$ are distinct.

2. Let H and K be subgroups of G with $HK = G$ and $H \cap K = 1$. Show that for any kH-module U the module $U \uparrow_H^G \downarrow_K^G$ is a direct sum of copies of the regular representation kK.

Solution. Observe that if $HK = G$ with $H \cap K = 1$ then also $H \cap K^g = 1$ for every $g \in G$, because such g can be written $g = kh$ with $h \in H$ and $k \in K$ (write $g^{-1} = h^{-1}k^{-1}$ and invert), so that $H \cap K^g = H \cap K^{kh} = H \cap K^h$. If there is some element $h^{-1}uh = v$ with $u \in K$ and $v \in H$, then $u = hvh^{-1} \in H$, so $u \in H \cap K = 1$. This shows that $H \cap K^g = 1$. Now Mackey's formula says

$$U \uparrow_H^G \downarrow_K^G = \bigoplus_{g \in [K \backslash G/H]} {}^g(U \downarrow^H_{H \cap K^g}) \uparrow^K_{{}^gH \cap K}$$

Every summand is induced from 1, so is a free kK-module.
Another, perhaps simpler, approach is to show that there is only one (K, H)-double coset in this situation, which means that we don't have to check that $H \cap K^g = 1$ for every $g \in G$.

3. Let k be a field. Show by example that it is possible to find a subgroup H of a group G and a simple kG-module U for which $U \downarrow_H^G$ is not semisimple.
Solution. The 2-dimensional reflection representation V of S_3 is simple in characteristic 2, and restricts to $\langle(1, 2)\rangle$ as a representation in which the element of order 2 acts as a matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is the regular representation of C_2, which is not semisimple.

4. Let H be a subgroup of G and V an RH-module. Show that if V can be generated by d elements as an RH-module then $V \uparrow_H^G$ can be generated by d elements as an RG-module.
Solution. One approach is to say that for V to be generated by elements $v_1, \ldots, v_d$ is equivalent to the existence of a surjection $RH^d \to V$ in which the ith coordinate vector maps to v_i. Inducing to G we get a map $RG^d = (RH^d) \uparrow_H^G \to V \uparrow_H^G$ which is surjective because tensor product is right exact. This means $V \uparrow_H^G$ can be generated by d elements.

5. Let H be a subgroup of G.
(a) Write $\overline{H} = \sum_{h \in H} h$ for the sum of the elements of H, as an element of RG. Show that $RG \cdot \overline{H} \cong R \uparrow_H^G$ as left RG-modules. Show also that $RG \cdot \overline{H}$ equals the fixed points of H in its action on RG from the *right*.
(b) More generally let $\rho : H \to R^\times$ be a 1-dimensional representation of H (that is, a group homomorphism to the units of R). Write $\tilde{H} := \sum_{h \in H} \rho(h)h \in RG$. Show that $RG \cdot \tilde{H} \cong \rho^* \uparrow_H^G$ as RG-modules.
Solution. For each element $g \in G$ the support of $g\overline{H}$ is the coset gH so that distinct elements $g\overline{H}$ have disjoint support on the basis of group elements of RG. This means that $RG \cdot \overline{H}$ has as a basis the elements $g\overline{H}$ where g ranges through a set of representatives of the cosets gH. This basis is permuted by G and the stabilizer of $\overline{H}$ is the subgroup H, so that $RG \cdot \overline{H} \cong R \uparrow_H^G$.
An element $\sum_{g \in G} a_g g$ of RG is fixed by H from the right if and only if, for all $h \in H$ we have $\sum_{g \in G} a_g g = \sum_{g \in G} a_g gh$, which means that the coefficients of g and gh are the same. This happens if and only if $\sum_{g \in G} a_g g$ can be written $x\overline{H}$ for some $x \in RG$. This means that $RG \cdot \overline{H}$ is as claimed.

(b) As in part (a), the support of $g\tilde{H}$ is the coset gH and so $RG\cdot\tilde{H}$ has as a basis the elements $g\tilde{H}$ where g ranges through a set of representatives of the cosets gH, this is a permutation module, the stabilizer of the space $R\tilde{H}$ is H and $RG\cdot\tilde{H}\cong R\tilde{H}\uparrow_H^G$ as RG-modules. We examine the representation of H on $R\tilde{H}$ and find that, for $x\in H$ we have $x\sum_{h\in H}\rho(h)h=\sum_{h'\in H}\rho(x^{-1}h')h'=\rho(x^{-1})\sum_{h\in H}\rho(h)h$, writing $h'=xh$ for a moment in the middle, and this is the representation ρ^*.

6. Let k be any field, and g any element of a finite group G.
(a) If $K\leq H\leq G$ are subgroups of G, V a kH-module, and W a kK-module, show that $({}^gV)\downarrow^{{}^gH}_{{}^gK}\cong {}^g(V\downarrow^H_K)$ and $({}^gW)\uparrow^{{}^gH}_{{}^gK}\cong {}^g(W\uparrow^H_K)$. [This allows us to put conjugation before, between, or after restriction and induction in Mackey's formula.]
(b) If U is any kG-module, show that $U\cong {}^gU$ by showing that one of the two mappings $U\to {}^gU$ specified by $u\mapsto gu$ and $u\mapsto g^{-1}u$ is always an RG-module isomorphism. [Find which one of these it is.]
Solution. (a) For the restriction formula, ${}^gV$ is the set V with ${}^gh\in {}^gH$ acting as ${}^gh\cdot v=hv$. If we restrict this to ${}^gK$ the same formula applies to both sides of the equation, so the two modules are the same (and not just isomorphic). For the induction formula we define a map $R{}^gH\otimes_{R{}^gK}{}^gW\to {}^g(kH\otimes_{kK}W)$ by ${}^gh\otimes w\mapsto h\otimes w$, and another map in the opposite direction ${}^gh\otimes w\leftarrow h\otimes w$. We have to check these are well defined on the tensor products, and they commute with the action of ${}^gH$.
(b) The map $U\to {}^gU$ given by $u\mapsto g^{-1}u$ is a vector space isomorphism, and it commutes with the action of G because if $x\in G$ then $xu\mapsto g^{-1}xu$ and $x\cdot g^{-1}u=(g^{-1}xg)g^{-1}u=g^{-1}xu$ is the same.

7. (Artin's Induction Theorem) Let $\mathbb{C}^{\mathrm{cc}(G)}$ denote the vector space of class functions on G and let $\mathcal{C}$ be a set of subgroups of G that contains a representative of each conjugacy class of cyclic subgroups of G. Consider the linear mappings

$$\mathrm{res}_{\mathcal{C}}:\mathbb{C}^{\mathrm{cc}(G)}\to\bigoplus_{H\in\mathcal{C}}\mathbb{C}^{\mathrm{cc}(H)}$$

and

$$\mathrm{ind}_{\mathcal{C}}:\bigoplus_{H\in\mathcal{C}}\mathbb{C}^{\mathrm{cc}(H)}\to\mathbb{C}^{\mathrm{cc}(G)}$$

whose component homomorphisms are the linear mappings given by restriction

$$\downarrow_H^G:\mathbb{C}^{\mathrm{cc}(G)}\to\mathbb{C}^{\mathrm{cc}(H)}$$

and induction

$$\uparrow_H^G:\mathbb{C}^{\mathrm{cc}(H)}\to\mathbb{C}^{\mathrm{cc}(G)}$$

(a) With respect to the usual inner product $\langle\ ,\ \rangle_G$ on $\mathbb{C}^{\mathrm{cc}(G)}$ and the inner product on $\bigoplus_{H\in\mathcal{C}}\mathbb{C}^{\mathrm{cc}(H)}$ that is the orthogonal sum of the $\langle\ ,\ \rangle_H$, show that $\mathrm{res}_{\mathcal{C}}$ and $\mathrm{ind}_{\mathcal{C}}$ are the transpose of each other. [For this we have to realize that if $\alpha:V\to W$ is a linear map then the usual transpose of α is a linear map $\beta : W \to V$ satisfying $\langle \alpha(v), w\rangle_W = \langle v, \beta(w)\rangle_V$ where $\langle -,-\rangle_V$ and $\langle -,-\rangle_W$ are the standard inner products on V and W defined with respect to given bases of V and W. A transpose may be defined using any pair of inner products like this.]

(b) Show that $\mathrm{res}_{\mathcal{C}}$ is injective.

[Use the fact that $\mathbb{C}^{\mathrm{cc}(G)}$ has a basis consisting of characters, that take their information from cyclic subgroups.]

(c) Prove Artin's induction theorem: In $\mathbb{C}^{\mathrm{cc}(G)}$ every character χ can be written as a rational linear combination

$$\chi = \sum a_{H,\psi}\psi\uparrow_H^G$$

where the sum is taken over cyclic subgroups H of G, ψ ranges over characters of H and $a_{H,\psi} \in \mathbb{Q}$.

[Deduce this from surjectivity of $\mathrm{ind}_{\mathcal{C}}$ and the fact that it is given by a matrix with integer entries. A stronger version of Artin's theorem is possible: there is a proof due to Brauer which gives an explicit formula for the coefficients $a_{H,\psi}$; from this we may deduce that when χ is the character of a $\mathbb{Q}G$-module the ψ that arise may all be taken to be the trivial character.]

(d) Show that if U is any $\mathbb{C}G$-module then there are $\mathbb{C}G$-modules P and Q, each a direct sum of modules of the form $V\uparrow_H^G$ where H is cyclic, for various V and H, so that $U^n \oplus P \cong Q$ for some n, where U^n is the direct sum of n copies of U.

Solution (a) For this we have to check that for all $H \in \mathcal{C}$ we have $\langle \chi\downarrow_H^G, \psi\rangle_H = \langle \chi, \psi\uparrow_H^G\rangle_G$. This is true when χ and ψ are characters because it is one of the formulas included in Frobenius reciprocity, and it is thus true for all class functions because characters span the class functions.

(b) Two class functions in $\mathbb{C}^{\mathrm{cc}(G)}$ are equal if and only if their values on all elements $g \in G$ are equal, which is implied by their restrictions to cyclic subgroups $\langle g\rangle$ being equal for all $g \in G$.

(c) The transpose of an injective map is surjective, so $\mathrm{ind}_{\mathcal{C}}$ is surjective. [The fact that the tranpose is surjective is familiar for the usual transpose of matrices. In terms of bilinear forms, if $\beta : W \to V$ is not surjective then we can find $0 \neq v \in V$ so that $\langle v, \beta(w)\rangle_V = 0$ for all $w \in W$. Thus $\langle \alpha(v), w\rangle_W = 0$ for all $w \in W$ and, by non-degeneracy of the inner product, $\alpha(v) = 0$. If α is injective, we obtain $v = 0$, which is a contradiction.]

Each space $\mathbb{C}^{\mathrm{cc}(G)}$ has a basis consisting of the characters of irreducible modules, and with respect to these basis the matrices of $\mathrm{res}_{\mathcal{C}}$ and $\mathrm{ind}_{\mathcal{C}}$ have integer entries. This implies that $\mathrm{ind}_{\mathcal{C}}$ is surjective as a map of the $\mathbb{Q}$-vector spaces with these bases, and thus the character of any module can be written as required.

(d) Writing each coefficient $a_{H,\psi}$ as $b_{H,\psi}/n$ if it is positive, and $-b_{H,\psi}/n$ if it is negative, where n and the $b_{H,\psi}$ are integers, we get $U^n \oplus P \cong Q$ where P is the direct sum of $V_\psi^{b_{H,\psi}}$ where the b are negative and Q is the direct sum of $V_\psi^{b_{H,\psi}}$ where the b are positive. Here V_ψ is a $\mathbb{C}G$-module with character ψ. This is because both sides have the same character.