Beginnings

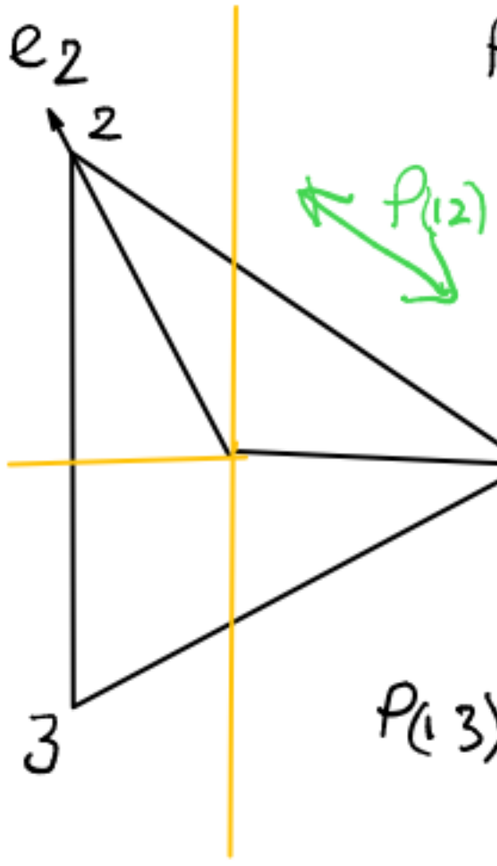
What is a group representation?

A representation G on
is: for each $g \in G$
a linear map $\rho_g : V \longrightarrow V$ so that

Let G be a group
V a vector space
- a vector space
- an abelian group
- an R-module

$\rho_{gh} = \rho_g \rho_h$ always
and $\rho_1 = \mathrm{id}$.

Example G = symmetries of $\triangle \cong S_3 \cong D_6$



$$\rho_{(12)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\rho_{(123)} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\rho_{(13)} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \quad \rho_{(23)} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

What is the matrix representing (1,3,2)?

A $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ B $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ C $\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ ✓

Group representations using matrices.

Taking a basis for V we get a matrix ρ_g for each $g \in G$.

Group representations as homomorphisms.

Definition. A representation of G is homomorphism

$$\rho : G \longrightarrow \mathrm{Aut}(V)$$

e.g. if V is a vector space we have a homom.

$$\rho : G \to GL(V).$$

If a group G acts on a topological space X then

G acts $H_n(X)$ $n \geq 0$.

$N' \trianglelefteq N$, N/N' is abelian

If N/H is abelian then $N' \subseteq H$.

If N is a normal subgroup of G then

G acts by conjugation as automorphisms of N, and hence as automorphisms of N/N', giving a representation.

Conjugation: if $g \in G$ $n \in N$

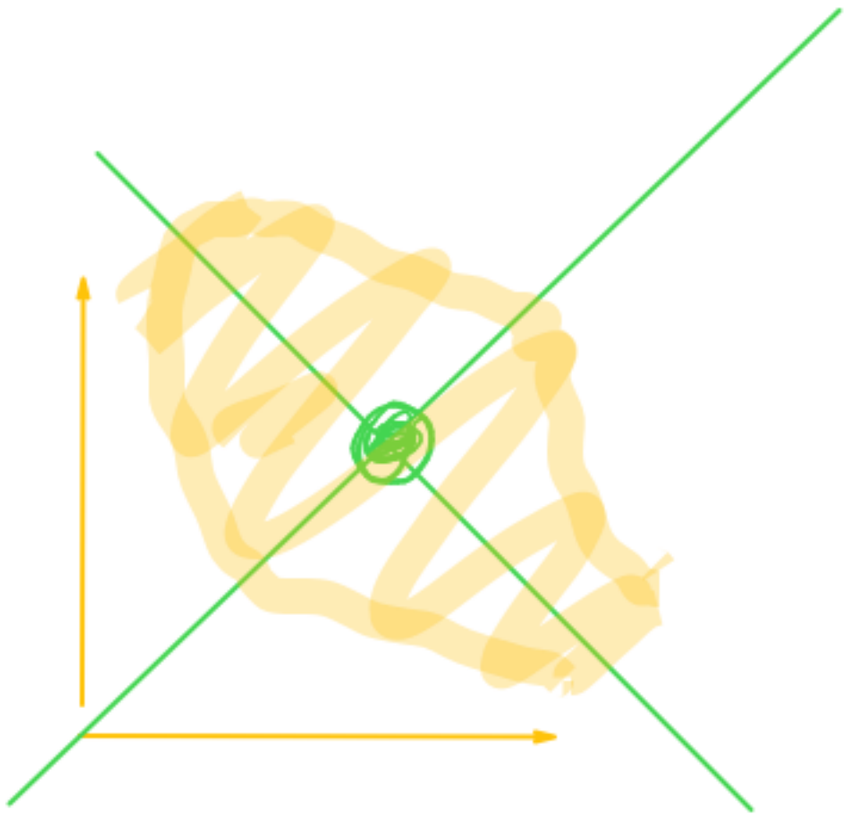
then $c_g(n) := gng^{-1}$

$c_g : N \to N$ giving

$\rho_g : N/N' \to N/N'$

$\rho_g(nN') = gng^{-1}N'$

$N' := \langle \underbrace{ghg^{-1}h^{-1}}_{[g,h]} \mid g, h \in N \rangle$



Green reflections generate a group $C_2 \times C_2$

Get a representation of $C_2 \times C_2$

Yellow region is a fundamental domain for the translations.

$\simeq$ set of orbit representatives

$\mathbb{R}^2 / \langle \text{translations} \rangle = T$

is

with opposite sides identified

= torus.

4 regions permuted by $C_2 \times C_2$. Which acts on $H_1(\text{torus}) = \mathbb{Z}^2$

$T \cong \mathbb{Z}^2$

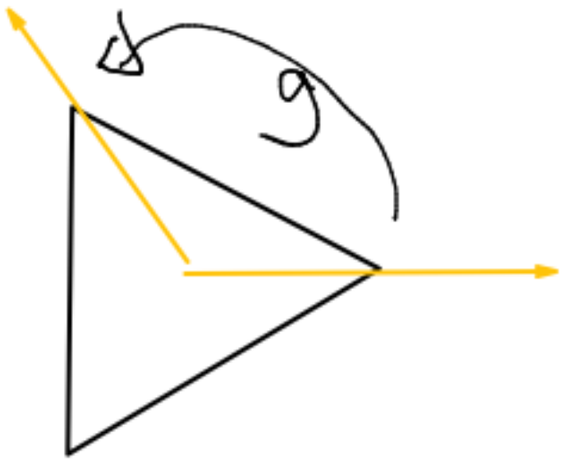
$C_2 \times C_2$ acts on this by conjugation.

These give representations

Representation of C_3

Pre-class Warm-up!!!

Let $C_3 = \langle g \rangle$ be a cyclic group of order 3, generated by an element g. It can be taken to be the 3 rotations that preserve an equilateral triangle:



Which of the following is a matrix that gives the action of g?

A $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ ✓

B $\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$ where $\omega = e^{\frac{2\pi i}{3}}$

C $\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}$ same ω.

D More than one of the above.

$\begin{bmatrix} \omega^2 \\ -1 \end{bmatrix}$ is an e-vector with e-value ω of A

$\begin{bmatrix} \omega \\ -1 \end{bmatrix}$ ..... ω^2.

giving B

This is the 'trivial representation'

Examples of representations

1. $V = R$, $\rho_g \cdot r = r \ \forall g \in G$.

Representations of cyclic groups

Write gr for $\rho_g r$

Let $G = C_n = \langle g \rangle$

We could take $V = \mathbb{C} = R$

ρ_g = multiplication by $\zeta = e^{\frac{2\pi i}{n}}$

$\cong [\zeta]$

$\rho_{g^t} = [\zeta^t]$

"W is invariant under G"

p is prime. $\mathbb{F}_p$ = field with p elements

A 2-dimensional representation of C_p over F_p

$$\rho(g) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \rho(g^t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

This $\rho: G \to GL_2(\mathbb{F}_p)$ is a homomorphism, ρ is a representation.

The subspace spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is invariant under G so gives a 1-dimensional subrepresentation

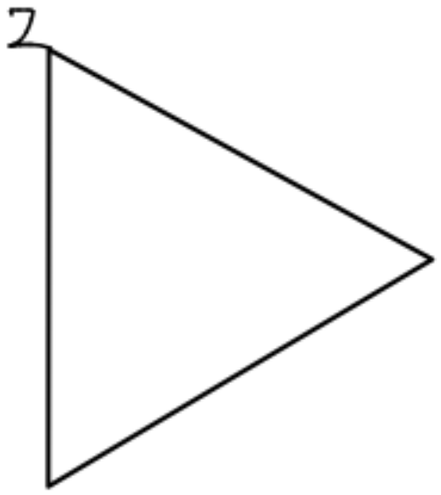
$\tau(g)\begin{bmatrix} 0 \\ 1 \end{bmatrix} := \rho(g)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We obtain a copy of the trivial representation.

Definition of sub representation. Given a representation $\rho: G \to GL(V)$ a subrepresentation is a repn $\tau: G \to GL(W)$ where $W \subseteq V$ is a submodule, with $\rho(x) W \subseteq W$ $\forall x \in G$

Subrepresentations

1. Subrepresentations of the representation of C_3 on 2-dimensional space.



This has two subrepresentations of dimension 1 over $\mathbb{C}$.

Over $\mathbb{R}$, there are no proper subrepresentations.
(They would have to 1-dimensional eigenspaces.)

This repn is not simple over $\mathbb{C}$.
It is simple over $\mathbb{R}$.

2. Subrepresentations of the representation of S_3 on 2-dimensional space.

There are no proper subrepresentations over $\mathbb{C}$ (or $\mathbb{R}$ or $\mathbb{Q}$).
Such representations would have dimn. 1, so would by e-spaces both for C_3 and for C_2

There are no such.

This repn is simple over $\mathbb{C}$

Definition. A representation with no non-trivial subrepresentations is called irreducible, or simple

The group algebra

Definition Let R be a commutative ring, G a finite group. The group ring is the abelian group

$RG :=$ set of formal R-linear combinations of elements of G

$=$ the set of symbols looking like $r_1 g_1 + r_2 g_2 + \cdots + r_t g_t$ for some t, $r_i \in R$, $g_i \in G$.

$= \sum_{i=1}^{t} r_i g_i$

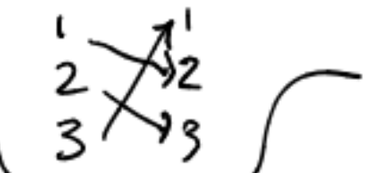
$(r_1 g_1) + (r_2 g_2) = (r_1 g_1 + r_2 g_2)$

$\cong$ free R-module with the elements of G as a basis.

Multiplication is determined by $(r_1 g_1)(r_2 g_2) := (r_1 r_2)(g_1 g_2)$

Conventions $f(x)$ $(1,2,3)$ means

Simplify $((1,2) - (1,2,3))(2,3)$



2nd 1st

A $(1,3,2) - (1,3)$

$2 \to 3 \to 3$

$3 \to 2 \to 1$

$(2,3,1)$

$=$

$(1,2,3)$

✓ B $(1,2,3) - (1,2)$

C $(2,3) - (1,3)$

D None of the above.

Example $G = S_3$

$(2(1,2) - (2,3))((1,3) + (1,2,3))$

$= 2(1\,3\,2) + 2(2,3) - (1,2,3) - (1\,3)$.

Proposition 1.1.5 Representations of G over R are the same thing as (unital) RG-modules.

What is a module?

What is the hardest thing about 1.1.5?

A. Understanding what the group algebra is.

B. Seeing that there should be any sort of connection at all between representations and modules.

C. Trying to understand what the algebraic symbols mean.

Why the group algebra?

Proposition. A module is simple if and only if it is generated by each of its non-zero elements.

Maschke's Theorem 1.2.1. Let V be a representation of the finite group G over a field F in which $|G|$ is invertible. Let W be an invariant subspace of V. Then there exists an invariant subspace W_1 of V such that V is the direct sum of W and W_1 as representations.

Proof.

Definition. A module that is a direct sum of simple submodules is said to be semi simple, or completely reducible.