Chapter 4: Constructions of representations.

We learn:
The complex characters of abelian groups are 1-dimensional (and conversely).
Characters of product groups.
Inflation, induction and restriction
Frobenius reciprocity
Symmetric and exterior powers

generalizes permutation repns.

Proposition 4.1.1. Over the complex numbers the cyclic group G = <g> of order n has precisely n simple characters given by

$$\chi_r(g^s) = \zeta_n^{rs}$$

where $\zeta_n = e^{2\pi i/n}$

Proof. There are n homomorphisms

$$G \longrightarrow \langle \zeta_n \rangle \subseteq \mathbb{C}^\times$$

$$g \longmapsto \zeta_n^r \qquad 0 \le r \le n-1$$

giving 1-dimensional representations of G, which are simple. They are distinct.

$\sum$ squares of degrees $= n = |G|$

so we have a complete list of simple repns of G over $\mathbb{C}$. $\square$

Characters of a product of groups.

Definition. Given groups G, H and repns V of G, W of H over R, we may make $V \otimes_R W$ into a repn of $G \times H$:

$(g,h)(v \otimes w) := gv \otimes hw$. Taking bases $v_1, \ldots, v_m$, $w_1, \ldots, w_n$ for V, W, (g,h) acts on $V \otimes W$ by the tensor product of the matrices by which g acts on V, h acts on W.

The characters multiply:

$$\chi_{V \otimes W}(g,h) = \chi_V(g)\,\chi_W(h).$$

Theorem 4.1.2. Let V_1,...,V_m and W_1,...,W_n be complete lists of the simple complex representations of G and H. Then $V_i \otimes W_j$ is a complete list of the simple representations of $G \times H$.

The list is complete!

Proof Ingredients:

Compute $\langle \chi_{V_i \otimes W_j}, \chi_{V_k \otimes W_l} \rangle$

$$= \frac{1}{|G \times H|} \sum_{\substack{g \in G \\ h \in H}} \overline{\chi_{V_i}(g)\chi_{W_j}(h)}\, \chi_{V_k}(g)\chi_{W_l}(h)$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)}\chi_{V_k}(g) \cdot \frac{1}{|H|} \sum_{h \in H} \overline{\chi_{W_j}(h)}\chi_{W_l}(h)$$

$$= \langle \chi_{V_i}, \chi_{V_k} \rangle \langle \chi_{W_j}, \chi_{W_l} \rangle$$

$$= \delta_{i,k}\,\delta_{j,l}$$

$\Rightarrow$ $V_i \times W_j$ is simple and distinct repns are distinct

The list is complete:

$\chi_{V_i \otimes W_j}(1) = \chi_{V_i}(1)\chi_{W_j}(1)$

$\sum$ squares of degrees = double sum $= |G||H|$.

Example: the character table of C_2 x C_2

$$\begin{array}{ccc} G & & H \\ \| & & \| \\ \langle g \rangle & & \langle h \rangle \end{array}$$

Char. table of G

	1	g
χ_1	1	1
χ_2	1	-1

of H

	1	h
ψ_1	1	1
ψ_2	1	-1

Char table of $G \times H$

	1	g	h	hg
$\chi_1 \otimes \psi_1$	1	1	1	1
$\chi_2 \otimes \psi_1$	1	-1	1	-1
$\chi_1 \otimes \psi_2$	1	1	-1	-1
$\chi_2 \otimes \psi_2$	1	-1	-1	1

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Corollary 4.1.4. The character table of G x H is the tensor product of the character table of G and the char. table of H.

Given G, H we get a repn of $G \times H$: $V \otimes W$

Take $G = H$ to get a repn of $G \times G$ on $V \otimes W$

If $R = \mathbb{C}$ then $V \otimes W$ is simple for $G \times G$ if $V \Delta W$ are W.

Before: we consider the homom

$$\delta: G \longrightarrow G \times G$$
$$g \longmapsto (g, g)$$

Given a repn $V \otimes W$ of $G \times G$ we get a repn of G by composing with δ.

$V \otimes W$ might not be simple as a repn of G.

Question: How many conjugacy classes has the group $S_3 \times S_3$

A 3

B 4

C 6

D 9

E 12

F 18

G None of the above.

Theorem 4.1.5. TFAE

1. G is abelian
2. All simple complex representations of G have degree 1.

Question. The cyclic group of order 3 has real characters

	1	g	g^2
χ_1	1	1	1
χ_2	2	-1	-1

Write $\chi_2 \otimes \chi_2$ as a sum of simple real characters.

A $\chi_2 + \chi_2$

B $\chi_2 + \chi_1 + \chi_1$

C $\chi_1 + \chi_1 + \chi_1 + \chi_1$

D None of the above.

Inflating from a quotient, degree 1 representations of any finite group G.

Proposition 4.2.1 The degree 1 representations of any finite group G over any field

Induction from a subgroup

Definition.

Background on tensor products

Proposition 4.3.1 Let H be a subgroup of G, let V be an RH-module and let

be a list of the left closets G/H. Then