Chapter 5: theorems of Clifford and Mackey

5.1 Double cosets

Let $g \in G$

Definition. Let H, K be subgroups of G. The double coset

$$HgK = \{hgk \mid h \in H, k \in K\}.$$

Example. The (S_2,S_2)-double cosets in S_3.

$S_3 = \{(), (12), (13), (23), (123)(132)\}$

$S_2 = \{(), (12)\}$.

$H = K = S_2$.

Take $g = ()$.

$H()K = \{(), (12)\} = H(12)K$.

$H(13)K = H(13)\{(), (12)\}$

$= H\{(1,3), (123)\} = \{(), (12)\}\{(13), (123)\}$

$= \{(13), (123), (132), (23)\}$

$H(13) \quad H(123)$

$= H(123)K = H(132)K = H(23)K$

$S_3 = H()K \sqcup H(13)K$.

Pre-class Warm-up!!

The symmetric group S_4 acts on the set $\Omega = \{1,2,3,4\}$ in the usual way, and also has subgroups isomorphic to $A_4, D_8, S_3, C_2 \times C_2$

Is Ω isomorphic as a set with S_4 action to any of the following coset spaces?

A S_4 / A_4

B S_4 / D_8

C S_4 / S_3 ✓

D $S_4 / C_2 \times C_2$

2. Is this a boring question?

Example: $S_2 \backslash S_3 / S_2$:

$S_3/S_2 \cong \{1, 2, 3\}$

$= \{()S_2, (123)S_2, (132)S_2\}$

$S_2 \backslash (S_3/S_2) = \{1,2\} \sqcup \{3\}$

with representatives

$1 \in \{1, 2, 3\}$

$3 \in \{1, 2, 3\}$

of

(123) and $()$.

Proposition 5.1.1. Let H, K be subgroups of G.

1. Each (H,K)-double closet is a disjoint union of right cosets of H, and of left cosets of K.

2. The (H,K)-double cosets partition G.

3. The (H,K)-double cosets are in bijection with orbits H\(G/K) and with the orbits (H\G)/K.

Proof.

1. If $x \in HgK$ then $Hx \subseteq HgK$
b/c $x = hgk$ so $h_1 x = h_1 hgk \in HgK$.
Thus HgK is a union of cosets Hx
Cosets are disjoint.

2. If $x \in HgK$ then $x = hgk$ so
$HxK = HhgkK = HgK$.
so Hg_1K, Hg_2K are the same or are disjoint.

3. Here G/K is the set of left cosets gK of K in G. G permutes these by left multiplication. So does H.
An H-orbit is a set of left cosets of K of the form $\{hgK \mid h \in H\}$
The K-orbits in $(H\backslash G)/K$ are sets $\{HgK \mid k \in K\}$.

Double cosets HgK $\longrightarrow$ orbits $H\backslash(G/K)$ $\{hgK \mid h \in H\}$

Hg_1K $\longleftarrow$ H-orbit of g_1K □

Notation: $H\backslash G/K$ will denote both the set of double cosets $H\backslash(G/K)$ and $(H\backslash G)/K$, and

Double coset representatives. A representative of the double coset HgK is any $x \in G$ so that $HgK = HxK$.

$[H\backslash G/K]$ denotes a set of (H,K) double coset representatives.

Definition. A G-set is a set Ω permuted by G

Homomorphism of G-sets: a mapping $f: \Omega \to \Psi$ that preserves the group action $f(xw) = xf(w)$ always.

Isomorphism of G-sets: a mapping $f: \Omega \to \Psi$ for which $\exists\, f_1 : \Psi \to \Omega$, so that $f_1 f = 1_\Omega$ and $f f_1 = 1_\Psi$.

Theorem (orbit-stabilizer theorem +)

Let Ω be a G-set and $w \in \Omega$.
The orbit $Gw \cong G/\mathrm{Stab}_G(w)$
as G-sets
where $\mathrm{Stab}_G(w) = \{g \in G \mid gw = w\}$.
Proof $gw \longleftrightarrow g\,\mathrm{Stab}_G(w)$.

$G/H \cong G/K \iff H \sim K$
are conjugate subgroups in G.

$\Omega\downarrow_H^G$ is the H-set obtained by only looking at H.

Theorem. Let Ω be a transitive G-set and let $K = \mathrm{Stab}_G(w)$ for some element $w \in \Omega$
so $\Omega \cong G/K$

a. If H is a subgroup of G then

$$\Omega\downarrow_H^G \cong \bigsqcup_{g \in [H\backslash G/K]} Hgw \quad \text{as } H\text{-sets.}$$

b. The orbit $Hgw \cong HgK/K$ as left H-sets.
Here HgK is a right K-set, $(HgK)/K$ is the set of K-orbits.

c. $HgK/K \cong H/(H \cap {}^gK)$ As left H-sets.
where ${}^gK := gKg^{-1}$

Proof a. Hgw is an H-orbit of Ω
When is $Hgw = Hg_1w$? i.e.
$g_1w \in Hgw$, i.e. $w \in g_1^{-1}Hgw$
$\exists\, h \in H$, $w = g_1^{-1}hgw$, $g_1^{-1}hg \in K$.
$hg \in g_1K$ $hgK \subseteq g_1KK = g_1K$
$HgK \subseteq Hg_1K$ i.e $HgK = Hg_1K$.
This means orbits Hgw $\longleftrightarrow$ double cosets HgK.

Theorem. Let Ω be a transitive G-set and let $K = \mathrm{Stab}_G(\omega)$ for some element $\omega \in \Omega$

a. If H is a subgroup of G then

$\Omega\downarrow_H = \bigsqcup_{g\omega \in [H\backslash\Omega]} Hg\omega$ as H-sets

b. The orbit $Hg\omega \cong HgK/K$ as left H-sets.

c. $HgK/K \cong H/(H \cap {}^gK)$ As left H-sets.

b. $hg\omega \longmapsto hgK$ is a mapping

$\Omega \longmapsto$ left K-cosets in HgK.

$hg\omega \longleftarrow hgK$

$\|$

$hgk\omega \longleftarrow hgk$

is a well defined inverse map.

$Hg\omega \longleftarrow HgK/K$

The left H-action is preserved.

c. $\mathrm{Stab}_H(gK) = \{h \in H \mid hgK = gK\}$

i.e. $g^{-1}hg \in K$, $h \in gKg^{-1}$ $= H \cap {}^gK$.

Pre-class Warm-up

1. Let S_4 act on $\Omega = \{1,2,3,4\}$.
How many orbits are there when Ω is restricted to S_3?

A 1

B 2 " S_3 has orbits $\{1,2,3\}, \{4\}$

C 3

D 4

2. How many (S_3,S_3)-double cosets are there in S_4? 2

3. Which was more difficult: question 1 or question2?

Q 1

Q 2

Definition. If H is a subgroup of G, g is an element of G, and V is a representation of H, we define. a representation ${}^gV$ of ${}^gH$ as follows:

${}^gV = V$ as a R-module with action of ${}^gH$

${}^gh \cdot v := hv \quad h \in H$

If $x \in {}^gH$ so $x^g = g^{-1}xg \in H$

then $x \cdot v := x^g v$.

If $y \in G$ $c_y : H \longrightarrow {}^gH$

$c_y(h) = yhy^{-1}$

is group homomorphism

If $\rho : G \to GL(V)$ is the repn

${}^gV$ is obtained as

$$G \xrightarrow{c_{g^{-1}}} G \xrightarrow{\rho} GL(V).$$

Proposition. In the decomposition

$$V\uparrow_H^G = \bigoplus_{g \in [G/H]} g \otimes V$$

$g \otimes V$ is a representation of ${}^gH$ isomorphic to ${}^gV$.

Proof. The left action of ${}^gh = ghg^{-1} \in {}^gH$ on $g \otimes_{RH} V$ is

$${}^gh(g \otimes V) = ({}^gh g) \otimes V$$
$$= ghg^{-1}g \otimes v = gh \otimes v$$
$$= g \times hv.$$

Identifying $g \otimes V \longleftrightarrow V$

$g \otimes v \longleftrightarrow v$

gives an isomorphism

$g \otimes V \cong {}^gV$ of ${}^gH$-modules.

Pre-class Warm-up!

Let H and K be subgroups of G, and consider the coset xK for some x in G. In the action on the left cosets G/K, is the stabilizer $Stab_H(xK)$

equal to

A $H^x \cap K$

B ${}^xH \cap K$

C $H \cap {}^xK$ ✓

D $H \cap K^x$

E None of the above.

Notation ${}^xH = xHx^{-1}$

$Stab_H(xK) = \{h \in H \mid hxK = xK\}$

$x^{-1}hxK = K$, $x^{-1}hx \in K$

$h \in xKx^{-1}$, $h \in H \cap {}^xK$

Theorem 5.2.1 (Mackey formula). Let H,K be subgroups of G and V a representation for K. Then

$$V\uparrow_K^G\downarrow_H^G \cong \bigoplus_{g\in[H\backslash G/K]} \left(g(V\downarrow^K_{H^g\cap K})\right)\uparrow^H_{H\cap {}^gK}$$

Proof. $V\uparrow_K^G = \bigoplus_{g\in[G/K]} g\otimes V$

Restrict left action to H.

$$V\uparrow_K^G\downarrow_H^G = \bigoplus_{H\text{-orbits of } HgK} \left(\bigoplus_{xK\subseteq HgK} x\otimes V.\right)$$

HgK of cosets $gK\in G/K$

as RH-modules.

$$= \bigoplus_{g\in[H\backslash G/K]}\ g\otimes\left(\bigoplus_{x\in[H/H\cap {}^gK]} x\otimes V.\right)$$

$$= \bigoplus_{g\in[H\backslash G/K]} \left({}^gV\right)\uparrow^H_{H\cap {}^gK}$$

Next time.

Theorem 5.2.1 (Mackey formula). Let H,K be subgroups of G and V a representation for K. Then

$$(V\uparrow_K^G)\downarrow_H^G \cong \bigoplus_{g\in[H\backslash G/K]} \left(g\left(V\downarrow^K_{H^g\cap K}\right)\right)\uparrow^H_{H\cap {}^gK}$$

as RH-modules.

Proof $V\uparrow_K^G = \bigoplus_{x\in[G/K]} x\otimes V$

where G permutes the subspaces $x\otimes V$ the same way as it permutes the cosets G/K.

Restrict the action to H. The orbits of H on G/K biject with double cosets HgK

It permutes the spaces $g\otimes V$ the same way it permutes the cosets gK.

$\bigoplus_{x\in[G/K]} x\otimes V$ is an H-orbit of these subspaces

$x\in HgK$

It is a direct sum of subspaces permuted by H, so is induced from

$\mathrm{Stab}_H(g\otimes V) = H\cap {}^gK = \mathrm{Stab}_H(gK)$

As a representation of $H\cap {}^gK$,

$$g\otimes V \cong g\left(V\downarrow^K_{H^g\cap K}\right)$$

Put this together.

$$\bigoplus_{x\in[G/K]} x\otimes V \cong \left(g\left(V\downarrow^K_{H^g\cap K}\right)\right)\uparrow^H_{H\cap {}^gK}$$

$x\in HgK$

$\mathrm{Stab}_H(gK) = \{h\in H \mid hgK = gK\}$

$g^{-1}hgK = K$, $h\in gKg^{-1}$ $h\in H\cap {}^gK$.

Example: permutation modules

Permutation modules are direct sums of induced modules $R\uparrow_K^G$ where R is the trivial module.

Restricting to $H \leq G$ we get a permutation module $(G/K)\downarrow_H$

$$= \coprod_{g \in [H\backslash G/K]} H / H \cap {}^gK .$$

$$R\uparrow_K^G\downarrow_H = \bigoplus_{g\in[H\backslash G/K]} \left({}^g(R\downarrow^G_{H^g\cap K})\right)\uparrow^G_{H\cap {}^gK}$$

Here ${}^g(R\downarrow^G_{H^g\cap K}) = R$ for $H\cap {}^gK$

so $R\uparrow_K^G\downarrow_H = \bigoplus_{g\in[H\backslash G/K]} R\uparrow^H_{H\cap {}^gK}$.

The S_4-set $\{1,2,3,4\}$

$\cong S_4/S_3$

We see $(R[S_4/S_3])\downarrow_D \cong R\uparrow^D_{D\cap S_3}$

$= R\uparrow^D_{C_2}$ ← non-central.

$\langle (14)(23), (1234) \rangle$

Question. Let D = < (1,2), (1,2,3,4) > be a dihedral subgroup of S_4 of order 8. How many (D,S_3) double cosets are there in S_4?

A 1 ✓

B 2

C 3

D >3

(D, S_3) double cosets in S_4

$\leftrightarrow$ orbits of D on S_4/S_3

$=$

$\{1,2,3,4\}$

How many orbits does D have on $\{1,2,3,4\}$?

Application of Mackey's formula and double cosets.

Definition. A group G permuting a set Ω is n-transitive if and only if

G is transitive on n-tuples of distinct elements of Ω.
$\Leftrightarrow$ G is transitive and $\mathrm{Stab}_G(\omega)$ is $(n-1)$ transitive on $\Omega - \{\omega\}$.

Examples.

S_n is n-transitive on $\{1,\dots,n\}$

A_n is $n-2$ transitive on $\{1,\dots,n\}$ if $n \geq 3$.

Proposition 5.2.3. Let G permute a set Ω Then

$$\mathbb{C}\Omega \cong \mathbb{C} \oplus V$$

for some module V. If $V \neq 0$ then V is simple if and only if G acts 2-transitively on Ω. In that case V is not the trivial representation.

Proof. $\mathbb{C}\Omega = \mathbb{C} \iff |\Omega| = 1$.

Suppose $|\Omega| \geq 2$. so $V \neq 0$.

Both sides of the $\iff$ fail unless G-transitive

Suppose $\Omega \cong G/K$ for some subgroup K.

$$\mathbb{C}\Omega \cong \mathbb{C}\uparrow_K^G$$

Write 1 = trivial character.

$$\langle \chi_{\mathbb{C}\uparrow_K^G}, \chi_{\mathbb{C}\uparrow_K^G} \rangle$$

$$= \langle 1\uparrow_K^G, 1\uparrow_K^G \rangle$$

$$= \langle 1, 1\uparrow_K^G \downarrow_K^G \rangle$$

$$= \langle 1, \sum_{g \in [K\backslash G/K]} 1\uparrow_{K\cap gK}^K \rangle_K$$

$$= \sum_{g \in [K\backslash G/K]} \langle 1, 1 \rangle_{K\cap gK}$$

$$= |K\backslash G/K|.$$

$[= 1 + \#\text{orbits of } \cancel{\mathrm{Stab}_G(\omega)}\ K]$

$= 2 \iff G$ is 2-transitive.

Thus $\langle \chi_V, \chi_V \rangle = 1$ so χ_V is simple.

Pre-class Warm-up!

True or false?:

Over any field K, if G is 2-transitive on Ω then $K\Omega = K \oplus V$ where either $V = 0$ or V is simple.

Another question: True or false:

Mackey's theorem only works over an algebraically closed field of characteristic 0.

Review of some things:

If H is a subgroup of G, g is an element of G and V is a representation of H we get a representation

${}^gV$ of ${}^gH = gHg^{-1}$

where ${}^gV = V$ as an R-module

and if ${}^gh \in {}^gH$, $v \in V$ then

${}^gh \cdot v := hv$

in ${}^gV$ in V

If H is a normal subgroup of G then

How easy is it to see that:
If V is a simple representation of H then ${}^gV$ is a simple representation of ${}^gH$.

A representation W of G has the form $V\uparrow_H^G$ if and only if
W is the direct sum of subspaces permuted by G with H being the stabilizer of one of them.

Clifford's theorem.

Theorem 5.3.1 (Weak form of Clifford's theorem)
Let k be any field, U a simple kG-module and N a normal subgroup of G. Then

$U\downarrow_N^G$

is semisimple as a kN-module.

Question: where was the hypothesis that N is normal used?

Theorem 5.3.1 (Clifford's theorem)
Let k be any field, U a simple kG-module and N a normal subgroup of G. We may write

$$U\downarrow_N^G = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$$

where the S_i are nonisomorphic simple kN-modules.

a. G permutes the homogeneous components transitively,
b. a_1 = a_2 = … and dim S_1 = dim S_2 = …

c. If $H = \mathrm{Stab}_G(S_1^{a_1})$ then $U \overset{N}{\cong} S_1^{a_1}\uparrow_H^G$ as kG-modules.

Theorem 5.3.3. Over an algebraically closed field k, if G is abelian then every simple kG-module has dimension 1.

Proof. Let S be a simple kG-module, let g be in G. Let S_λ be an eigenspace of g with eigenvalue λ.
All elements h of G commute with g so $hS_\lambda = S_\lambda$.
Thus S_λ is a kG-submodule of S, so $S_\lambda = S$.
It follows: every element of G acts by scalar multiplication, so dim S = 1.

Review of things about p-groups.

Every non-trivial p-group G has center $Z(G) \neq 1$.

Lemma. Every non-abelian p-group has a normal abelian subgroup that is not central.

Corollary 5.3.4. Let k be an algebraically closed field and G a p-group, for some prime p. Then every simple kG-module has the form

$$U\uparrow_H^G$$

where U is a 1-dimensional module for some subgroup H.

Proof. Induction on |G|.
Let S be a simple representation with kernel N.
By induction, we may assume that N = 1, so that G embeds in GL(S).
If G is abelian, we are done.
Otherwise, let A be a normal, non-central abelian subgroup.
Restrict to A and use Clifford:

$$S\downarrow_A = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r} \qquad \text{Put } V = S_1^{a_1}$$

Here $S = V\uparrow_K^G$ where $K = \mathrm{Stab}_G(S_1^{a_1})$

V must be a simple kK-module.

If $K \neq G$ we use induction: $V = U\uparrow_H^K$, $\dim U = 1$
$S = (U\uparrow_H^K)\uparrow_K^G = U\uparrow_H^G$.

We eliminate K = G: in this case
A is abelian so dim S_1 = 1. Elements of A act as scalar multiplication, which commutes with the action of G. Since G embeds in GL(S), A would be central, a contradiction.