Chapter 6: Representations in characteristic p: beginnings

Proposition 6.1.1. Let k be a field of characteristic p . Then

$$kC_{p^n} \cong k[X]/(X^{p^n}) \text{ as } k\text{-algebras}$$

Let $C_{p^n} = \langle g \rangle$.

Proof. We define the algebra homomorphism $\phi : k[x] \longrightarrow kC_{p^n}$ by $\phi(X) = 1-g$.

Then $\phi(X^{p^n}) = (1-g)^{p^n}$

$$= 1 - \binom{p^n}{1}g + \binom{p^n}{2} \quad + (-1)^{p^n}\binom{p^n}{p^n}g^{p^n}$$

$$= (1 - g^{p^n}) = (1-1) = 0.$$

$(X^{p^n}) \subseteq \ker\phi$. We get an induced homom $k[x]/(X^{p^n}) \to kC_{p^n}$

Theorem 6.1.2. Let k be a field of characteristic p. Every $k[x]/(X^{p^n})$-module is the direct sum of cyclic modules

$$U_r = k[X]/(X^r)$$

where 1 ≤ r ≤ p^n
Each module U_r has a unique composition series.

It is onto b/c it has

$\phi(1) = 1$

$\phi(X) = 1-g$

$\phi(X^2) = (1-g)^2 = 1-2g+g^2$

etc.

in its image, which is spanned by $1, g, g^2, \ldots g^{p^n-1}$

Image $= kC_{p^n}$. Compare dimensions

Both p^n. Get an isomorphism.

Pre-class Warm-up

Let k be a field. How many distinct ideals does the ring k[x]/ (x^3) have?

A 1

B 2

C 3

D 4

E 3|k| where |k| is the cardinality of k

F None of the above.

Any ideal of $k[x]/(x^3)$ has the form $I/(x^3)$ where I is an ideal of $k[x]$, containing (x^3)

Such I have the form (f) where where f divides x^3.

Such f are scalar multiples of $1, x, x^2, x^3$

4 possibilities.

Definition. A module M is cyclic if it can be generated by a single element.

Example simple modules are cyclic.

A module M is indecomposable if it is not possible to write $M = M_1 \oplus M_2$ with $M_1 \neq 0 \neq M_2$.

Examples 1. Simple modules are indecomposable.

2. The $\mathbb{F}_p C_p$-module where g acts as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ on $\mathbb{F}_p^2 = V$, $C_p = \langle g \rangle$.

The only submodules of V are 0, V, $\mathbb{F}_p \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A module with a unique simple submodule (= simple socle) is indecomposable.

Theorem 6.1.2. Let G be a cyclic group of order p^n and k a field of characteristic p. There are ~~p~~ p^n isomorphism types of indecomposable kG-modules, each cyclic and uniserial

uniserial = unique composition series

= submodules are linearly ordered by inclusion.

Proof. We show the indecomposable kC_{p^n} modules have the form

$$U_r = kC_{p^n} / ((1-g)^r) \quad 1 \leq r \leq p^n$$

kC_{p^n} modules are the same as $k[X]/(X^{p^n})$-modules

= $k[X]$-modules on which X^{p^n} acts as 0.

These have the form $k[X]/(f_1) \oplus \cdots \oplus k[X]/(f_t)$ for $f_i \in k[X]$ (modules for a P.I.D.)

X^{p^n} acts as 0 means $f_i \mid X^{p^n}$, $(f_i) = (X^r)$ $1 \leq r \leq p^n$.

Indecomposable modules are $k[X]/(X^r)$ $1 \leq r \leq p^n$.

The submodules are $(X^s)/(X^r)$ $s \leq r$. Unique simple submodule, so indecomposable. Also uniserial. $\square$

Jordan blocks. Diagrams

More about what $U_r = k[X]/(X^r)$ looks like.

It has a k-basis

$$1+(X^r),\ X+(X^r),\ \dots,\ X^{r-1}+(X^r).$$

X acts via

$$\begin{bmatrix} 0 & & & & \\ 1 & 0 & & \multicolumn{2}{c}{\Large 0} \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ \multicolumn{2}{c}{\Large 0} & & 1 & 0 \end{bmatrix}$$

which is a Jordan block.

As a module for kC_{p^n}

$X \leftrightarrow 1-g \qquad g \leftrightarrow 1-X$

acts via

$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix}$$

not quite a Jordan block.

If I had done $X \leftrightarrow 1+g$

We get $g = X-1$

$$\begin{bmatrix} -1 & & & \\ 1 & -1 & & 0 \\ & & \ddots & \\ & & & -1 \\ 0 & & 1 & -1 \end{bmatrix}$$

Picture of the basis

$$\begin{array}{cl} \bullet & 1+(X^r) \\ X \downarrow & \\ \bullet & X+(X^r) \\ X \downarrow & \\ \bullet & \\ X \downarrow & \\ \vdots & \\ \bullet & \\ X \downarrow & \\ \bullet & \end{array}$$

nodes $\leftrightarrow$ basis elements.

1. Every finite dimensional module is a direct sum of indecomposable modules.
2. Krull-Schmidt theorem

Up to isomorphism, decompositions as $\oplus$ indecomposables are unique for representation over a field.

Proposition 6.2.1. Let k be a field of characteristic p and G a p-group. The only simple kG-module is the trivial module.

Fun

Fact: Every maximal subgroup of a finite p-group is normal of index p.

Proof. Let S be simple.
Induction on $|G|$. Take a max. subgp $N \leq G$. It is normal.
$S\downarrow_N$ is semisimple by Clifford.
by induction N acts as 1 on each simple, so N acts as 1 on S.
Thus S is a module for $k[G/N]$, inflated to G.

Now G/N is cyclic, which only has k as its unique simple module. So $S \cong k$. $\square$

Another fact: for each prime p, every finite group G has a unique largest normal p-subgroup. This subgroup is denoted

Corollary 6.2.2. Let k be a field of characteristic p and G a finite group. The common kernel of the action of G on all the simple kG-modules is $O_p(G)$. Thus the simple kG-modules are precisely the simple $k[G/O_p(G)]$-modules, inflated to G.

Examples.