Chapter 6: Representations in characteristic p: beginnings

Proposition 6.1.1. Let k be a field of characteristic p . Then

$$kC_{p^n} \cong k[X]/(X^{p^n}) \text{ as } k\text{-algebras}$$

Let $C_{p^n} = \langle g \rangle$.

<u>Proof</u>. We define the algebra homomorphism $\phi : k[x] \to kC_{p^n}$ by $\phi(X) = 1-g$.

Then $\phi(X^{p^n}) = (1-g)^{p^n}$

$= 1 - \binom{p^n}{1} g + \binom{p^n}{2} \cdots + (-1)^{p^n}\binom{p^n}{p^n} g^{p^n}$

$= (1 - g^{p^n}) = (1-1) = 0.$

$(X^{p^n}) \subseteq \ker \phi$. We get an induced homom $k[x]/(X^{p^n}) \to kC_{p^n}$

Theorem 6.1.2. Let k be a field of characteristic p. Every $k[x]/(X^{p^n})$-module is the direct sum of cyclic modules

$$U_r = k[X]/(X^r)$$

where $1 \le r \le p^n$
Each module U_r has a unique composition series.

It is onto b/c it has

$\phi(1) = 1$

$\phi(X) = 1-g$

$\phi(X^2) = (1-g)^2 = 1-2g+g^2$

etc.

in its image, which is spanned by $1, g, g^2, \ldots, g^{p^n-1}$

Image $= kC_{p^n}$. Compare dimensions both p^n. Get an isomorphism.

Pre-class Warm-up

Let k be a field. How many distinct ideals does the ring k[x]/ (x^3) have?

A 1

B 2

C 3

D 4

E 3|k| where |k| is the cardinality of k

F None of the above.

Any ideal of $k[x]/(x^3)$ has the form $I/(x^3)$ where I is an ideal of $k[x]$, containing (x^3)

Such I have the form (f) where where f divides x^3.

Such f are scalar multiples of $1, x, x^2, x^3$

4 possibilities.

Definition. A module M is cyclic if it can be generated by a single element.

Example simple modules are cyclic.

A module M is indecomposable if it is not possible to write $M = M_1 \oplus M_2$ with $M_1 \neq 0 \neq M_2$.

Examples 1. Simple modules are indecomposable.

2. The $\mathbb{F}_p C_p$-module where g acts as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ on $\mathbb{F}_p^2 = V$, $C_p = \langle g \rangle$.
The only submodules of V are 0, V, $\mathbb{F}_p \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. A module with a unique simple submodule (= simple socle) is indecomposable.

Theorem 6.1.2. Let G be a cyclic group of order p^n and k a field of characteristic p. There are ~~pn~~ p^n isomorphism types of indecomposable kG-modules, each cyclic and uniserial

uniserial = unique composition series
= submodules are linearly ordered by inclusion.

Proof. We show the indecomposable kC_{p^n} modules have the form
$U_r = kC_{p^n} / ((1-g)^r)$, $1 \le r \le p^n$
kC_{p^n} modules are the same as $k[X]/(X^{p^n})$-modules
= $k[X]$-modules on which X^{p^n} acts as 0.
These have the form $k[X]/(f_1) \oplus \cdots \oplus k[X]/(f_t)$
for $f_i \in k[X]$ (modules for a P.I.D.)
X^{p^n} acts as 0 means $f_i \mid X^{p^n}$,
$(f_i) = (X^r)$ $1 \le r \le p^n$.
Indecomposable modules are $k[X]/(X^r)$ $1 \le r \le p^n$.
The submodules are $(X^s)/(X^r)$ $s \le r$. Unique simple submodule, so indecomposable. Also uniserial. $\square$

Jordan blocks. Diagrams

More about what $U_r = k[X]/(X^r)$ looks like.

It has a k-basis

$1+(X^r),\ X+(X^r),\ \ldots,\ X^{r-1}+(X^r)$.

X acts via
$$\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix}$$
which is a Jordan block.

As a module for kC_{p^n}

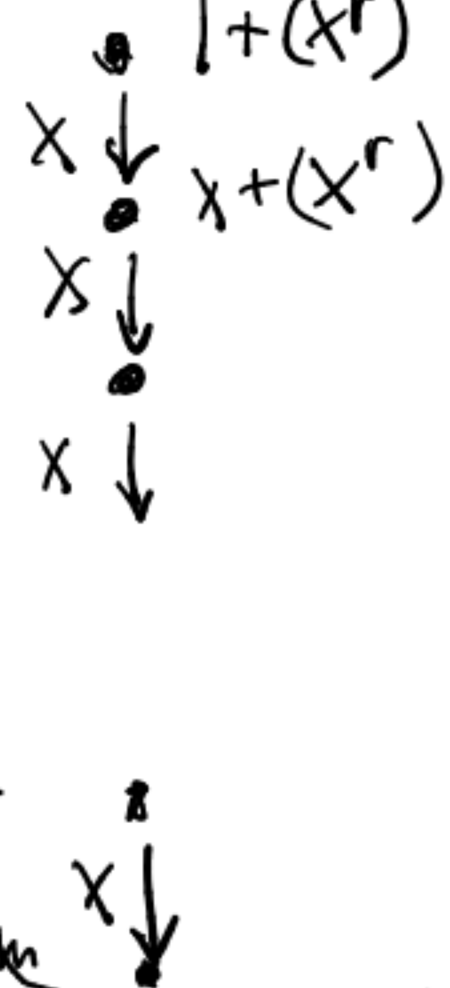
$X \leftrightarrow 1-g \qquad g \leftrightarrow 1-X$

acts via
$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix}$$
not quite a Jordan block.

If I had done $X \leftrightarrow 1+g$

We get $g = X-1$
$$\begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}$$

Picture of the basis



nodes $\leftrightarrow$ basis elements.

1. Every finite dimensional module is a direct sum of indecomposable modules.
2. Krull-Schmidt theorem

Up to isomorphism, decompositions as $\oplus$ indecomposables are unique for representation over a field.

Proposition 6.2.1. Let k be a field of characteristic p and G a p-group. The only simple kG-module is the trivial module.

Fun
Fact: Every maximal subgroup of a finite p-group is normal of index p.

Proof. Let S be simple. Induction on $|G|$. Take a max. subgp $N \leq G$. It is normal.

$S\downarrow_N$ is semisimple by Clifford. by induction N acts as 1 on each simple, so N acts as 1 on S.

Thus S is a module for $k[G/N]$, inflated to G.

Now G/N is cyclic, which only has k as its unique simple module. So $S \cong k$. $\square$

Pre-class Warm-up

In the first question of HW1 we proved that a 2-dimensional representation V of S_3 over F_2, given by certain matrices, is simple.
What is the dimension over F_2 of

$$\mathrm{Hom}_{\mathbb{F}_2 S_3}(\mathbb{F}_2 S_3, V) \quad ?$$

A 0

B 1

C 2 ✓

D 4

E 6

F 12

$\mathrm{Hom}_{\mathbb{F}_2 S_3}(\mathbb{F}_2 S_3, V) \longleftrightarrow V$

$\phi \mapsto \phi(1)$

$(x \mapsto xv) \leftarrow\!\!\shortmid \; v$

Comp'n factors of $\mathbb{F}_2 S_3$ are $\mathbb{F}_2, \mathbb{F}_2, V, V$. So these are the only simple $\mathbb{F}_2 S_3$-modules.

Or: $\mathbb{F}_2 S_3 = \mathbb{F}_2 \uparrow_1^{S_3}$

$$\mathrm{Hom}_{\mathbb{F}_2 S_3}(\mathbb{F}_2 \uparrow_1^{S_3}, V) \cong \mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, V\downarrow_1)$$

has dim 2.

Deduce: $\mathbb{F}_2 S_3$ has only two simple modules $\mathbb{F}_2$, V.
Briefly: dim Hom( ) = 2 and $\mathrm{End}_{\mathbb{F}_2 S_3}(V, V)$ has dim 1 (Schur lemma+)
$\Rightarrow$ $\mathbb{F}_2 S_3$ has ≥ 2 copies of V as composition factors. Also $\mathbb{F}_2$ is a composition factor. There remains a dim'n 1 c.f. which is a rep'n of $S_3/S_3' = C_2$ which only has $\mathbb{F}_2$ as a simple module.

Another exciting fact: for each prime p, every finite group G has a unique largest normal p-subgroup. This subgroup is denoted $O_p(G)$

(Also $\exists$ smallest normal $H \trianglelefteq G$ with G/H is a p-group. Such $H = O^p(G)$.)

Proof. If $H, K \trianglelefteq G$ are they p-subgroups then

$HK \trianglelefteq G$

and HK is a p-group. b/c

$HK/H \cong K/K \cap H$ is p-gp

and so is H.

$O_p(G)$ = product of all normal p-subgroups.

Examples

$O_2(A_4) = \langle (12)(34), (13)(24) \rangle$

$O_2(S_4)$ = some group.

Corollary 6.2.2. Let k be a field of characteristic p and G a finite group. The common kernel of the action of G on all the simple kG-modules is O_p(G). Thus the simple kG-modules are precisely the simple k[G/O_p(G)]-modules, inflated to G.

Proof. Let S be a simple kG-module

Then $S\downarrow_{O_p(G)}$ is simple by Clifford. $O_p(G)$ is a p-group, so has k as its only simple module.

Deduce $O_p(G)$ acts as 1 on S

Thus S is a repn of $G/O_p(G)$. inflated to G.

Let $H = \{h \in G \mid h \cdot v = v \ \forall v \in S \ \forall \text{ simples } S\}$

Then $H \lhd G$. We show H has only elements of order p^n, $n \geq 0$.

Because $H \supseteq O_p(G)$, we get $H = O_p(G)$.

If not, $\exists\, h \in H$ of order prime to p.

$kG\downarrow_{\langle h \rangle}$ is the direct sum of its kG-composition factors restricted $\langle h \rangle$.

h acts as 1 on each of these.

h acts as 1 on kG

It follows $h = 1$, b/c no other element fixes kG.

Examples.

Examples

1. $G = A_4$ $k = \mathbb{F}_4$.

Simple kG-modules

= simple modules for

$A_4/$Klein 4-group.

$\cong C_3$.

which has 3 simple repns.

k_1, k_ω, k_{ω^2}

where $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$

These are also the simple reps of A_4 over $\mathbb{F}_4$.

2. $G = S_4$ $k = \mathbb{F}_2$

$G/O_2(G) = S_4/$Klein 4-group.

$\cong S_3$.

Simple $\mathbb{F}_2 S_4$-modules

= simple $\mathbb{F}_2 S_3$-modules

$= \{\mathbb{F}_2, V\}$.

6.3 Radicals, socles, the augmentation ideal.

These definitions work for an arbitrary (non-commutative) ring A with a 1.

Definition. If U is an A-module we put

$\mathrm{Rad}(U) = \bigcap \{ M \mid M$ is a maximal submodule of $U\}$

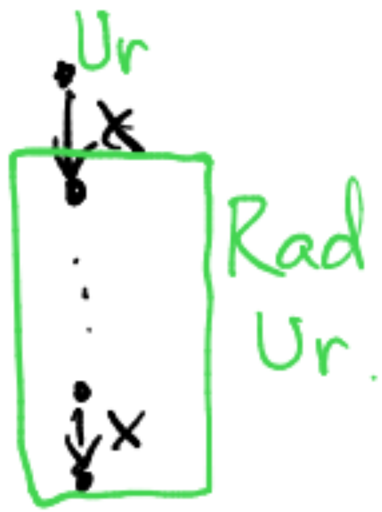
If the intersection is empty we put Rad(U) = U.

If U is Noetherian then the $\bigcap$ is non-empty.

Example: $k[X]/(X^n)$ has n indecomposable modules

$U_r = (X^{n-r})/(X^n) =$

$\mathrm{Rad}(U_r) \cong U_{r-1}$



Lemma 6.3.1 Let U be an A-module.

a. Suppose M_1, … ,M_n are maximal submodules of U. There is a subset I of {1, … ,n} such that

$$U / (M_1 \cap \dots \cap M_n) \cong \bigoplus_{i \in I} U/M_i$$

which is semisimple.

b. If U has DCC on submodules then U / Rad U is a semisimple module, and Rad U is the unique smallest submodule of U with this property.

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Proof. Let $I \subseteq \{1, \dots ,n\}$ be maximal so that

$$U/\bigcap_{i\in I} M_i \cong \bigoplus_{i\in I} U/M_i$$

We show $\bigcap_{i\in I} M_i = M_1 \cap \cdots \cap M_n$ and argue by contradiction. If not, there would exist M_j with $\bigcap_{i\in I} M_i \nsubseteq M_j$

The homomorphism $f : U \to \left(\bigoplus_{i\in I} U/M_i\right) \oplus U/M_j$ has kernel

$$M_j \cap \bigcap_{i\in I} M_i$$

It will suffice to show f is subjective.

Note that $\bigcap_{i\in I} M_i + M_j = U$.

If x is in U we can write x = y + z where

$$y \in \bigcap_{i\in I} M_i, \quad z \in M_j$$

Let $g : U \to U/\bigcap_{i\in I} M_i \oplus U/M_j$

g is surjective because g(y) = (0, x+M_j), $g(z) = (x+\bigcap_{i\in I} M_i, 0)$

It follows that f is surjective because f = g composed with the isomorphism that identifies $U/\bigcap_{i\in I} M_i$ with $\bigoplus_{i\in I} U/M_i$

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