Chapter 7: Projective modules

It may be useful to know the standard equivalent definitions of a projective module. A is a rng

Proposition 7.1.3.

TFAE for an A-module P.

1. P is a direct summand of a free module A^n

2. In every diagram

where α is an epimorphism

$\exists\ \gamma : P \to A$ so that $\beta = \alpha\gamma$

3. $\mathrm{Hom}(P, -)$ is exact

4. Every epimorphism $A \to P$ is split epi.

Importance:

Every module is a homomorphic image of a projective module

Projective modules achieve the longest radical series.

Pre-class Warm-up!

Consider the element

e = () + (1,2,3) + (1,3,2)

In the group ring F_2S_3
(where F_2 is the field with 2 elements, S_3 is the symmetric group).

Is e idempotent?

A Yes

B No

(Further question: is e idempotent in the group ring over the integers Z ?)

$$\left(\sum_{g\in G} g\right)^2 = |G| \sum_{g\in G} g$$

$$\left(\frac{1}{|G|}\sum_{g\in G} g\right)^2 = \frac{1}{|G|}\left(\sum_{g\in G} g\right)$$

$$e^2 = 3e$$

Proposition 7.2.1. Let A be a ring. The decompositions of the regular representation as a direct sum of submodules

$${}_AA = A_1 \oplus \cdots \oplus A_r$$

biject with expresssions

$$1 = e_1 + \cdots + e_r$$

as a sum of orthogonal idempotents, in such a way that $A_i = Ae_i$

The summand A_i is indecomposable if and only if the corresponding idempotent is primitive.

Idempotents $e^2 = e$

orthogonal: $ef = 0 = fe$

primitive $e = e_1 + e_2$ with $e_1 e_2 = 0 = e_2 e_1$ then either $e_1 = 0$ or $e_2 = 0$.

Example: $M_{2,2}(\mathbb{Q}) = A$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is primitive

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not primitive

$$A\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \right\}$$

$$A\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \right\}$$

non-free projectives (isomorphic to each other here.)

$$A = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \right\} \oplus \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \right\}.$$

Proof. Given $A = A_1 \oplus \cdots \oplus A_r$ write $1 = e_1 + \cdots + e_r$, $e_i \in A_i$.

Given $1 = e_1 + \cdots + e_r$ define $A_i = Ae_i$. Then $A = Ae_1 \oplus \cdots \oplus Ae_r$ etc. □

Example. G = S_3 , R = F_4 = {0, 1, ω, ω^2}.

The elements

$$e_1 = () + (1,2,3) + (1,3,2)$$

$$e_2 = () + \omega(1,2,3) + \omega^2(1,3,2)$$

$$e_3 = () + \omega^2(1,2,3) + \omega(1,3,2)$$

are orthogonal idempotents that sum to 1. ✓

We show that each of the RGe_i is indecomposable, and that RGe_1 has two trivial composition factors, the other RGe_i are both the simple 2-dimensional module.

What we know about RG-modules:

All 1-dimensional modules are modules for $C_2 \cong S_3/\langle(1,2,3)\rangle$, inflated to S_3.

$\mathbb{F}_4 C_2$ only has the trivial rep. as 1-dim'l rep. so $\mathbb{F}_4 S_3$ only has one 1-dim'l repn. We have seen a 2-D-repn. It's simple

Use the fact that $1+\omega+\omega^2 = 0$.

We could calculate

$e_2^2 = e_2$

$e_1 e_2 = e_2 e_1 = 0 \quad e_2 e_3 = e_3 e_2 = 0$

Thus

$\mathbb{F}_4 S_3 = \mathbb{F}_4 S_3 e_1 \oplus \mathbb{F}_4 S_3 e_2 \oplus \mathbb{F}_4 S_3 e_3$

is a direct sum of projective modules.

I claim: Each $\mathbb{F}_4 S_3 e_i$ has basis (over $\mathbb{F}_4$): e_i, $(1,2)e_i$

e.g. take $i=1$. These elements are $() + (123) + (132)$,

$(12) + (13) + (23)$. which are fixed by $\langle(1,2,3)\rangle$, interchanged by $(1,2)$. They span an $\mathbb{F}_4 S_3$-submodule.

Deduce $\mathbb{F}_4 S_3 e_1$ = permutation module on the cosets of $\langle (1,2,3) \rangle$

$$\mathbb{F}_4 S_3 e_1 = \mathbb{F}_4 \uparrow_{\langle (123) \rangle}^{S_3}$$

$\langle (123) \rangle$ acts trivially on it.

$$\mathbb{F}_4 S_3 e_1 = \mathbb{F}_4 C_2 \text{ inflated to } S_3 \text{ via } S_3 \to C_2.$$

This has unique composition series with two trivial factors. It is indecomposable.

What about $\mathbb{F}_4 S_3 \cdot \left(() + \omega(123) + \omega^2 (132) \right)$?

Verify It has basis $e_2, (12)e_2$.

$(123) \cdot e_2 = \omega^2 e_2$

$(123) \cdot (12) e_2 = \omega (12) e_2$

$e_2, (12)e_2$ are e-vectors for $(1,2,3)$ with e-values ω^2, ω.

(12) interchanges them.

$\mathbb{F}_4 S_3 e_2$ has no 1-dim submodule for $\mathbb{F}_4 S_3$. It is simple.

$\mathbb{F}_4 S_3 e_2 \cong \mathbb{F}_4 S_3 e_3$

$x e_2 \longleftrightarrow x (12) e_3$

is an isomorphism.

We conclude

$$\mathbb{F}_4 S_3 \cong 1 \oplus 2 \oplus 2$$

The only simple $\mathbb{F}_4 S_3$-modules are trivial = 1 and 2 = simple 2-dim'l repn.

Pre-class Warm-up!

Let R be a field of characteristic 2.
Is the following permutation module projective?

$$R\uparrow^{S_3}_{\langle(1,2,3)\rangle}$$

A Yes ✓

B No

$$RS_3 = RS_3e_1 \oplus RS_3e_2 \oplus RS_3e_3$$

$$e_1 = () + (1,2,3) + (1,3,2)$$

RS_3e_1 has basis $\{e_1, (1,2)e_1\}$ and is projective. It is a permutation module $\mathrm{Stab}_G(e_1) = \langle(1\,2\,3)\rangle$

Extra question:
Is the following (different) permutation module projective?

$$R\uparrow^{S_3}_{\langle(1,2)\rangle} = R \oplus \text{2-dim'l module.}$$

A Yes

B No ✓

In fact

$$RS_3 = \begin{smallmatrix}1\\1\end{smallmatrix} \oplus 2 \oplus 2$$

The trivial module $1 = R$ is not projective, b/c $\begin{smallmatrix}1\\1\end{smallmatrix}$ is indecomposable

$$R\uparrow^{S_3}_{\langle(1\,2\,3)\rangle} \cong RS_3e_1$$

Facts about projective modules for finite dimensional algebras A. (Theorems 7.3.8 and 7.3.9)

Let P be an indecomposable projective module. Then Rad P is the unique maximal submodule of P, so that S_P := P / Rad P is a simple module.
For every simple S, $\exists$ projective Q with $Q/\mathrm{Rad}\,Q \cong S$

P is determined up to isomorphism by S_P
We get a bijection {simples} $\leftrightarrow$ {indecomposable projectives}

Every indecomposable projective is a direct summand of A. By the Krull-Schmidt theorem.

Examples p-groups in char p.

$\mathbb{F}_2 D_8$

$S_{\mathbb{F}_2 D_8}$ is simple.

Theorem. Over an algebraically closed field R,

$$A \cong \bigoplus P^{\dim S_P}$$ where $S_P = P/\mathrm{Rad}(P)$

'Proof' If $A = \bigoplus P^{n_P}$ compute

$$n_P = \dim \mathrm{Hom}_A(A, S_P)$$
$$= \dim S_P \quad \square$$

If P is indecomposable projective and $P/\mathrm{Rad}(P) \cong S$ we say P is the projective cover of S.

Example. G = S_3, R is a field of

Done

$G = D_{2n}$, n odd.

D_{2n} has a cyclic subgroup C_n

and if R is large enough (with a primitive n^{th} root ζ of 1) then

$RC_n \cong \bigoplus$ n 1-dim'l reps.

$$= \bigoplus RC_n e_i$$

where $e_i = \frac{1}{n}(1 + \zeta^r g + \zeta^{2r} g^2 + \zeta^{3r} g^3 + \cdots)$

$C_n = \langle g \rangle$.

These are orthogonal idempotents.

$RD_{2n} = RD_{2n} e_1 \oplus RD_{2n} e_2 \oplus \cdots \oplus RD_{2n} e_n$

Each $RD_{2n} e_i$ has basis

e_i, ye_i where $y \in D_{2n} - C_n$

has order 2.

$RD_{2n} e_1 = R\uparrow_{C_n}^{D_{2n}} = 1$

Each $RD_{2n} e_i$ is indecomposable

If $i \neq 1$ RD_{2n} is a simple 2-dimensional.

$G = A_4$ $p = 2$ has a cyclic subgroup $\langle (1,2,3) \rangle = C_3$

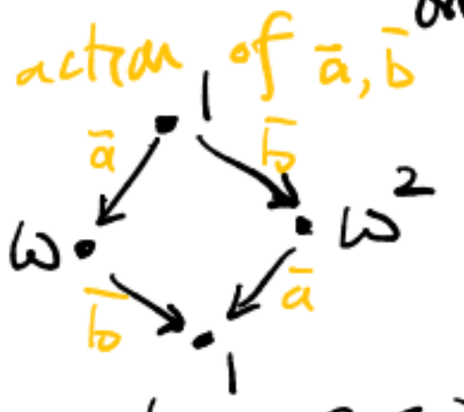
RC_3 has idempotents e_1, e_2, e_3 as with RS_3.

$RA_4 = RA_4e_1 \oplus RA_4e_2 \oplus RA_4e_3$

If $e_1 = () + (123) + (132)$

$RA_4e_1 = R\uparrow_{C_3}^{A_4}$ has dimension 4.

$\langle (12)(34), (13)(24) \rangle$ acts regular on $R\uparrow_{C_3}^{A_4}$.

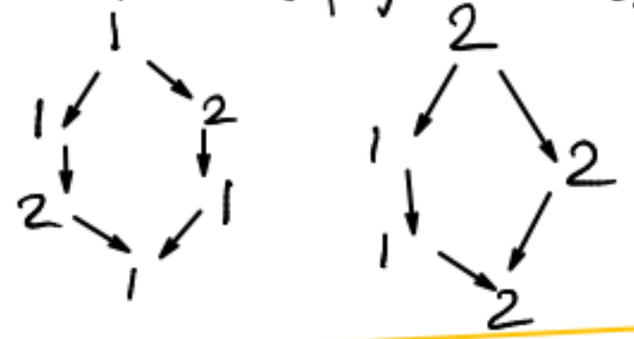
$R\uparrow_{C_3}^{A_4} =$ 

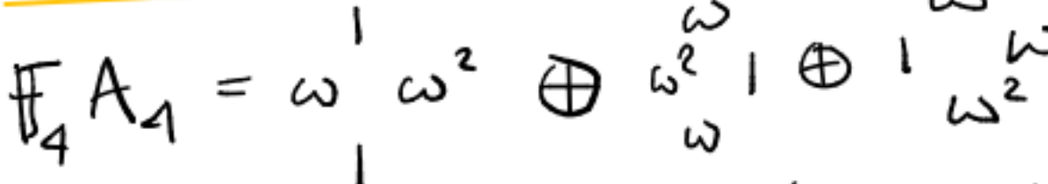
where $\omega^3 = 1$

The basis vectors are e-vectors for C_3 with e-value shown.

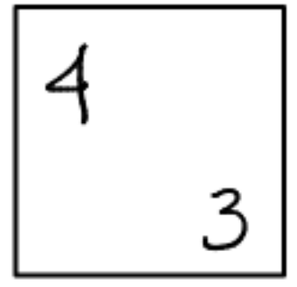
Note that over $\mathbb{F}_4$, A_4 has 3 simple reprs because $A_4/O_2(A_4) \cong C_3$

$G = S_4$, $p = 2$. Simples are reps of indecomposable projectives. $S_4/O_2(S_4) \cong S_3$.



$\mathbb{F}_4A_4 =$ 

shows the indecomposable projectives.



The Cartan matrix:
Question: what is the (1,2)-entry?

Pre-class Warm-up

How many (isomorphism classes of) representations of degree (=dimension) 1 does the alternating group A_4 have over the field F_3 with 3 elements?

A 1 ✓

B 2

C 3

D 4

E > 4

Distribution of correct answers:

1-dimensional reprs
= 1-D reprs of A_4/A_4' inflated to A_4.
$\| C_3$

$\#_3 C_3$ has one 1-dim'l. repn.

The Cartan matrix

$C = (c_{ST})$

where c_{ST} = multiplicity of the simple module S as a composition factor of the indecomposable projective P_T (where T is simple)

P_T has the property $P_T / \mathrm{Rad} P_T \cong T$

Examples.

1. S_3 in characteristic 2
2. p-groups in characteristic p.
3. A_4 in characteristic 2.
4. S_4 in characteristic 2.

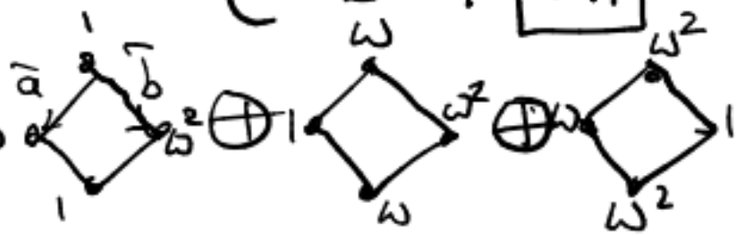
$\mathbb{F}_4 S_3 = 1 \oplus 2 \oplus 2$ $\quad P_1 = \begin{smallmatrix}1\\1\end{smallmatrix}$, $P_2 = 2$

$\{\text{simples}\} = \{1, 2\}$

$$C = \begin{array}{c|cc} & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{array}$$

2. p-groups in characteristic have k as their only simple, 1

$P_k = kG$ $\quad C = \boxed{|G|}$

3. $\mathbb{F}_4 A_4 =$ 

$$C = \begin{array}{c|ccc} & 1 & \omega & \omega^2 \\ 1 & 2 & 1 & 1 \\ \omega & 1 & 2 & 1 \\ \omega^2 & 1 & 1 & 2 \end{array}$$

More facts: remarkable for group rings.

- the Cartan matrix is symmetric

- each indecomposable projective has a simple socle

- projective modules are also injective, and vice-versa.

Example. S_4 in characteristic 2.
The simple kS_4-modules are the simple $kS_3 \cong k[S_4/V]$-modules, inflated.
Here $V = \{(), (12)(34), (13)(24), (14)(23)\}$
$= O_2(S_4)$
There are two such: 1, 2

Unbelievable fact: the indecomposable projective modules for kS_4 look like:

P_1

P_2

The Cartan matrix is

4	2
2	3

C is symmetric for group rings.

Question: what is the (1,2) entry of the Cartan matrix?

A 0

B 1

C 2

D 3

E 4

Amazing facts about projective modules

Lemma. Let H be a subgroup of G.

1. If V is a projective module for kG then $V{\downarrow}^G_H$ is a projective module for kH.

2. If W is a projective module for kH then $W{\uparrow}^G_H$ is a projective module for kG.

3. If V is a projective module for kG and M is any kG-module then $V \otimes_k M$ is projective.

Proof. 1. It suffices to consider the case when $V = kG$, because arbitrary projectives are summands of kG^n

$$kG{\downarrow}_{kH} = \bigoplus_{g \in [H\backslash G]} kHg$$

because $G = \bigsqcup_{g \in [H\backslash G]} Hg$

and $kH \cong kHg$ as left kH-modules

$$x \longleftrightarrow xg$$

2. Suffices: $W = kH$

$W{\uparrow}^G_H = kG \otimes_{kH} kH = kG$ is free hence projective.

3. Suffices: $V = kG = k{\uparrow}^G_1$.

Now $V \otimes_k M \cong (k{\uparrow}^G_1 \otimes_k M)$

$= (k \otimes_k M{\downarrow}^G_1){\uparrow}^G_1 = kG^{(\dim M)}$

is free, hence projective.

Pre-class Warm-up!!

What is the dimension of the radical of $\mathbb{F}_4 S_3$?

A 0

B 1 ✓

C 2

D 3

E 4

F 5

Do you think any of the following elements lies in the radical of $\mathbb{F}_4 S_3$?

A $() + (1,2) + (2,3) + (2,3) + (1,2,3) + (1,3,2)$ $\in \mathrm{Rad}\, \mathbb{F}_4 S_3$

B $() + (1,2)$

C $() + (1,2,3) + (1,3,2)$] Not in $\mathrm{Rad}\, \mathbb{F}_4 S_3$

$R \cdot \sum_{g \in G} g$ is a 2-sided ideal.

It is nilpotent: $\left(\sum_{g \in G} g\right)^2 = |G| \sum_{g \in G} g$

$= 0$

$\mathrm{Rad}\, \mathbb{F}_4 S_3 = R \cdot \sum_{g \in G} g$.

Theorem 8.4.1. Let k be a field of characteristic p. TFAE

$\exists K \trianglelefteq G, |G/K| = |G|_p$ = order of Sylow p-subgp.

1. G has a normal p-complement
2. For every simple kG-module S, the composition factors of the projective cover P_S are all isomorphic to S.
3. The composition factors of P_k are all isomorphic to k.

Proof (1) $\Rightarrow$ (3). We can write $kK = k \oplus W$ as kK-modules

$k = kK \cdot e_1$, $e_1 = \frac{1}{|K|} \sum_{g \in K} g$.

$kG = kG \cdot e_1 \oplus kG(1-e_1)$ as kG-modules which are projective.

$kGe_1 \cong k\uparrow_K^G$ (basis $\{ge_1, g \in [G/K]\}$)

$\cong k[G/K]$

- a group ring of a p-group in char p. All its composition factors are k.

Definition G has a normal p-complement means G as a normal subgroup $K \trianglelefteq G$ with $|G/K| = |G|_p$ = p-part of order of G,

$\Leftrightarrow |K| = |G|_{p'}$

e.g. S_3 $p=2$, A_4, $p=3$ have normal p-complements.

$\mathbb{F}_2 S_3 = 1 \oplus 2 \oplus 2$

We have a kG-projective all of whose comp factors are k. This is P_k.

(3) $\Rightarrow$ (2) Let S be a simple kG-module $P_k \otimes_k S$ is projective and its composition factors are all $k \otimes_k S \cong S$. The same is true of any direct summand. Such a summand is P_S.

(2) $\Rightarrow$ (3) is clear.

Theorem 8.4.1. Let k be a field of characteristic p.
TFAE
1. G has a normal p-complement
2. For every simple kG-module S, the composition factors of the projective cover P_S are all isomorphic to S.
3. The composition factors of P_k are all isomorphic to k.

(3) $\Rightarrow$ (1). Suppose all compn factors of P_k are $\cong k$.

Let $g \in G$ be any element of order prime to p.

$k\langle g\rangle$ is semisimple.

$P_k \downarrow_{\langle g\rangle}$ has a composition series with quotients $k \downarrow^{G}_{\langle g\rangle} = k$.

Therefore $P_k \downarrow_{\langle g\rangle} \cong k^n$

where $n = \dim P_k$. g acts trivially on P_k.

Let $K = \langle g \mid g \text{ has order prime to } p\rangle$

This is a normal subgroup and it acts trivially on P_k.

We show K contains no element of order p.

For such x, $P_k \downarrow^{G}_{\langle x\rangle}$ is projective

$k\langle x\rangle$ has only one indecomposable projective ($k\langle x\rangle$) and x does not act trivialy on it. x does not act trivially on P_k. No such x exists.

and $p \nmid |K|$. Thus $|G/K|$ is the order of a Sylow p-subgroup of G. Hence (1). $\square$

Examples.

1. S_3 (or D_{2n} when n is odd) in characteristic 2. Done that.
2. A_4 in characteristic 3.

A_4 has $V = \langle (12)(34), (13)(24) \rangle$ $\lhd A_4$ as a normal 3 complement

$\mathbb{F}_3 V$ has 4 idempotents.

$e_1, \dots, e_4$. $k = \mathbb{F}_3$

$$kA_4 = kA_4 e_1 \oplus kA_4 e_2 \oplus kA_4 e_3 \oplus kA_4 e_4.$$

$e_1 = \sum_{g \in V} g$.

$kA_4 e_1$ has basis

e_1, $(1,2,3)e_1$, $(1,3,2)e_1$ and has dimension 3.

$kA_4 e_1 \downarrow_{\langle (123) \rangle}$ is projective for $k\langle (123) \rangle$ so has dimension divisible by 3. Same is true for other summands, which are non-zero

Therefore $\dim kA_4 e_i = 3$ $\forall i$.

$(1,2,3)$ acts on $kA_4 e_i$ as $k\langle (1,2,3) \rangle$.

Thus $P_k = kC_3$ inflated to A_4 via $C_3 = A_4/V$.

kA_4 only has one 1-dim'l module. (It is repn of $C_3 = A_4/A_4'$.)

Other simples S have $\dim S > 1$.

$\dim P_S \leq 3$. All compn factors of P_S are S.

Deduce. Non-trivial simples have dim 3.

$kA_4 e_i$ is simple if $i \neq 1$

Each P_S occurs with multiplicity $\dim S$.

Thus $kA_4 e_2 \cong kA_4 e_3 \cong kA_4 e_4$ are isomorphic simples.

$$kA_4 = \vdots \oplus 3 \oplus 3 \oplus 3$$

as left kA_4-modules.

In fact

$$kA_4 \cong \mathbb{F}_3(x)/(x^3) \oplus M_3(\mathbb{F}_3)$$

as rings.

Also $\mathbb{F}_4 S_3 = \mathbb{F}_4[X]/(X^2) \oplus M_2(\mathbb{F}_4)$

as rngs.