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Commute Times for a Directed Graph using an Asymmetric Laplacian

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Abstract

The expected commute times for a strongly connected directed graph are related to an asymmetric Laplacian matrix as a direct extension to similar well known formulas for undirected graphs. We show the close relationships between the asymmetric Laplacian and the so-called Fundamental matrix. We give bounds for the commute times in terms of the stationary probabilities for a random walk over the graph together with the asymmetric Laplacian and show how this can be approximated by a symmetrized Laplacian derived from a related weighted undirected graph.

1 Introduction

The spectral analysis of undirected graphs has been studied extensively [1, 8, 10, 12, 14, 16, 25, 26, 27, 28, 34, 38], but fewer papers exist discussing directed graphs (digraphs) [4, 7, 9, 39]. In particular, the relationship between expected first transit/hitting times and round-trip commute times in a random walk, on the one hand, and spectral properties of the underlying graph on the other, has been studied mainly for undirected graphs. In this paper, we show that the round-trip commute times are closely related to certain asymmetric "Laplacian" matrices for strongly connected directed graphs in ways analogous to those known for undirected graphs. We show that one can approximate a strongly connected digraph by a related weighted undirected graph which shares some of the properties of the original digraph (e.g. connectivity, stationary probabilities), while only approximately inheriting others (e.g. first transit/hitting times and node centrality). This has applications in domains with asymmetric connections, such as wireless packet switching networks with low-powered units where link asymmetry is a widely observed phenomenon.

A directed graph, or digraph, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is a collection of vertices (or nodes) $i \in \mathcal{V} = \{1, \ldots, n\}$ and directed edges $(i \to j) \in \mathcal{E}$. One can, optionally, assign weights to each

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directed edge, thereby making it a so-called weighted digraph, or else a common edge weight of 1 to obtain an unweighted digraph. Algebraically, the digraph $\mathcal{G}$ can be represented by its $n \times n$ **adjacency matrix** $A = [a_{ij}]$, where $a_{ij} \neq 0$ is the weight on edge $(i \to j)$ and $a_{ij} = 0$ if $(i \to j) \notin \mathcal{E}$. A directed graph $\mathcal{G}$ is called **strongly connected** or a **strong digraph** if there is a path $i{=}\ell_0 \to \ell_1 \to \cdots \to \ell_{\kappa-1} \to \ell_\kappa{=}j$ for any pair of nodes i, j, where each link $\ell_{\iota-1} \to \ell_\iota$, $\iota = 1, \ldots, \kappa$, is an edge in the graph. In this paper, we focus entirely on strongly connected directed graphs.

A random walk over a graph can be modeled by a Markov chain with **probability transition matrix** $P = D^{-1}A$, where $D = \mathrm{Diag}(\mathbf{d}) = \mathrm{Diag}(A \cdot \mathbf{1})$ is the diagonal matrix of vertex out-degrees and $\mathbf{1}$ denotes the vector of all ones. Here we assume every node has at least one out-going edge, which can include self-loops. The associated **vector of stationary probabilities** is denoted by $\boldsymbol{\pi}$ and satisfies $\boldsymbol{\pi}^{\mathrm{T}} P = \boldsymbol{\pi}^{\mathrm{T}}$ and $\boldsymbol{\pi}^{\mathrm{T}}\mathbf{1} = 1$. If the graph is strongly connected, the associated Markov chain is irreducible, and all the entries of $\boldsymbol{\pi}$ are strictly positive by Perron-Frobenius theory [15, 19]. If the graph were undirected, the associated Markov chain would be reversible, and the vector of stationary probabilities would be a scalar multiple of the vector of vertex degrees: $\boldsymbol{\pi} = \mathbf{d}/(\mathbf{d}^{\mathrm{T}}\mathbf{1})$, where the denominator would be called the *volume* of the graph. Unfortunately, this relationship does not necessarily hold for digraphs. We denote by $\Pi = \mathrm{Diag}(\boldsymbol{\pi})$, the diagonal matrix of stationary probabilities, which is non-singular if the graph is strongly connected. These quantities have proven useful in the analysis of graphs and form the basis for this paper. For more details on Markov chains and their close relationships with graphs, the reader is referred to [20, 21, 29].

In this work, we examine a scaled "Laplacian," not necessarily symmetric and denoted simply by L, which is defined for a strongly connected directed graph or a strong digraph. In what follows, the words graphs and digraphs will be used strictly to mean strong digraphs, unless otherwise stated. Even though most of the derivations mimic known derivations for undirected graphs, not everything carries over from the world of undirected graphs to that of their directed counterparts. For example, the concept of "volume" of a graph and the metaphor of resistances of an electrical network [5, 11, 22] do not play the obvious central role in the derivations for directed graphs as they do for undirected graphs.

Our focus is on an asymmetric Laplacian ($L = \Pi(I - P)$) and its related matrices, which help illustrate parallels in the directed case to the well known properties defined for undirected graphs. In particular, we show the following for strongly connected directed graphs:

a. The average hitting times and round-trip commute times can be expressed in terms of the pseudo-inverse of this Laplacian.

b. The commute time is a distance measure for the vertices of a strongly connected directed graph.

c. There is a close relationship between the so-called Fundamental Matrix and the pseudo-inverse of the asymmetric Laplacian (L).

d. The commute times for a directed graph can be bounded in terms of the stationary probabilities and the eigenvalues of a diagonally scaled symmetrized graph Laplacian.

The rest of this paper is organized as follows. Section 2 gives some elementary necessary lemmas regarding the pseudo-inverse of matrices under rank-one changes. Section 3 compares the different Laplacians for directed graphs. Section 4 reviews the Fundamental Matrix and its relation to the pseudo-inverse of the Laplacian and to the matrices of expected hitting and commute times for a directed graph. Section 5 derives upper and lower bounds for the commute times in terms of the stationary probabilities together with the Fundamental Matrix and/or the diagonally scaled Laplacian. Section 6 shows how the Laplacian yields an indicator of node centrality based on average commute times for directed graphs in much the same way as for undirected graphs. Section 7 uses a simple example to show how treating a wireless network as a directed graph, which is more accurate, can yield a different result compared to the traditional analysis as an undirected graph.

2 The Pseudo-Inverse Under Small Rank Changes

The development in this paper makes use of several lemmas regarding general square matrices with nullity equal to 1, and their pseudo-inverses under small rank modifications. Here **nullity** is the dimension of the right null space.

Some notations warrant a mention here. The first two lemmas concern a general square irreducible matrix L such that $\text{nullity}(L) = 1$, and its Moore-Penrose pseudo-inverse $M = L^+$. By a simple singular value decomposition, one can see that $\text{nullity}(L) = 1 \Leftrightarrow \text{nullity}(M) = 1$. Recall that the *adjugate* of a matrix A, $\text{adj}(A)$ is the transpose of the matrix of cofactors of A: $[\text{adj}(A)]_{ij} = \det(A_{-j,-i})$, where $A_{-j,-i}$ denotes the $(n-1) \times (n-1)$ matrix formed from A by deleting row j and column i. We are now ready for the first of our lemmas.

Lemma 1 [35]. If A is a square matrix such that $\text{nullity}(A) = 1$, and $\mathbf{u}, \mathbf{v}$ are non-zero vectors such that $A\mathbf{u} = A^{\mathrm{T}}\mathbf{v} = \mathbf{0}$, then the adjugate of A is a rank-one matrix given as $\text{adj}(A) = \alpha \mathbf{u}\mathbf{v}^{\mathrm{T}}$, for some scalar α.

Proof. By [35], $\text{nullity}(A) = 1 \Rightarrow \text{rank}(\text{adj}(A)) = 1$, hence $\text{adj}(A) = \alpha \mathbf{x}\mathbf{y}^{\mathrm{T}}$ for some non-zero vectors $\mathbf{x}, \mathbf{y}$. Since $A \cdot \text{adj}(A) = \text{Det}(A) \cdot I = 0$, $\mathbf{x}$ must be in the right nullspace of A and hence is a non-zero multiple of $\mathbf{u}$. Likewise, $\mathbf{y}$ must be a non-zero multiple of $\mathbf{v}$.
□

Lemma 1 provides an easy way to compute the adjugate of a square matrix L with $\text{nullity}(L) = 1$. Computing the left and right nullspaces yields $\mathbf{u}, \mathbf{v}$, and computing one principal minor yields the scale-factor α. We remark that if $\text{nullity}(A) = 0$, then $\text{adj}(A) = \det(A) \cdot A^{-1}$, whereas if $\text{nullity}(A) > 1$, then $\text{adj}(A) = 0$ [35].

Lemma 2. Let $L = \begin{pmatrix} L_{11} & \mathbf{l}_{12} \\ \mathbf{l}_{21}^{\mathrm{T}} & l_{nn} \end{pmatrix}$ be an $n \times n$ irreducible matrix such that $\text{nullity}(L) = 1$. Let $M = L^+$ be the pseudo-inverse of L partitioned similarly and $(\mathbf{u}^{\mathrm{T}}, 1)L = 0$, $L(\mathbf{v}; 1) = 0$, where $\mathbf{u}, \mathbf{v}$ are $(n-1)$-vectors. Here, the operator ';' denotes vertical concatenation *à la*

Matlab. Then the inverse of the $(n-1)\times(n-1)$ matrix L_{11} exists and is given by

$$L_{11}^{-1} = X \stackrel{\text{def}}{=} (I_{n-1} + \mathbf{v}\mathbf{v}^{\mathrm{T}})M_{11}(I_{n-1} + \mathbf{u}\mathbf{u}^{\mathrm{T}}) = (I_{n-1}\,,\,-\mathbf{v}) \begin{pmatrix} M_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{21}^{\mathrm{T}} & m_{nn} \end{pmatrix} \begin{pmatrix} I_{n-1} \\ -\mathbf{u}^{\mathrm{T}} \end{pmatrix}, \tag{1}$$

where I_{n-1} denotes the $(n-1)\times(n-1)$ identity matrix.

Proof. Note that, $L_{11}\mathbf{v} + \mathbf{l}_{12} = \mathbf{0}$ and $\mathbf{u}^{\mathrm{T}}L_{11} + \mathbf{l}_{21}^{\mathrm{T}} = \mathbf{0}$, and $l_{nn} = -\mathbf{u}^{\mathrm{T}}\mathbf{l}_{12} = +\mathbf{u}^{\mathrm{T}}L_{11}\mathbf{v}$. Given $M = L^{+}$, the right annihilating vector for L is the left annihilating vector for M and viceversa, i.e. $(\mathbf{v}^{\mathrm{T}}, 1)M = 0$ and $M(\mathbf{u}; 1) = 0$.

Hence, $M_{11}\mathbf{u} + \mathbf{m}_{12} = \mathbf{0}$ and $\mathbf{v}^{\mathrm{T}}M_{11} + \mathbf{m}_{21}^{\mathrm{T}} = \mathbf{0}$, and $m_{nn} = -\mathbf{v}^{\mathrm{T}}\mathbf{m}_{12} = +\mathbf{v}^{\mathrm{T}}M_{11}\mathbf{u}$. Therefore, for the $(n-1)\times(n-1)$ matrix X we have the following form;

$$\begin{aligned} X &\stackrel{\text{def}}{=} I_{n-1} + \mathbf{v}\mathbf{v}^{\mathrm{T}})M_{11}(I_{n-1} + \mathbf{u}\mathbf{u}^{\mathrm{T}}) \\ &= M_{11} + M_{11}\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}}M_{11} + (\mathbf{v}^{\mathrm{T}}M_{11}\mathbf{u})\mathbf{v}\mathbf{u}^{\mathrm{T}} \\ &= M_{11} - \mathbf{m}_{12}\mathbf{u}^{\mathrm{T}} - \mathbf{v}\mathbf{m}_{21}^{\mathrm{T}} + m_{nn}\mathbf{v}\mathbf{u}^{\mathrm{T}} \\ &= (I_{n-1}\,,\,-\mathbf{v})M \begin{pmatrix} I_{n-1} \\ -\mathbf{u}^{\mathrm{T}} \end{pmatrix} \end{aligned} \tag{2}$$

We now show that L_{11} must have an inverse by contradiction. Suppose $L_{11}\mathbf{x} = 0$ such that $\mathbf{x} \neq 0$. Then $\mathbf{l}_{21}^{\mathrm{T}}\mathbf{x} = -\mathbf{u}^{\mathrm{T}}L_{11}\mathbf{x} = 0$ which means that $L(\mathbf{x}, 0)^{\mathrm{T}} = 0$. However, this would mean that we have a second right annihilating vector which is not a multiple of $(\mathbf{v}, 1)^{\mathrm{T}}$. This contradicts the initial assumption that nullity$(L) = 1$.

Let X be as defined above in equation (2). Multiplying on the left and right sides of X by L_{11} we get;

$$\begin{aligned} L_{11}XL_{11} &= L_{11}(I_{n-1}\,,\,-\mathbf{v})M \begin{pmatrix} I_{n-1} \\ -\mathbf{u}^{\mathrm{T}} \end{pmatrix} L_{11} &&= (L_{11}\,,\,\mathbf{l}_{12})M \begin{pmatrix} L_{11} \\ \mathbf{l}_{21}^{\mathrm{T}} \end{pmatrix} \\ &= \big(I_{n-1}\,,\,\mathbf{0}\big)\, LML \begin{pmatrix} I_{n-1} \\ \mathbf{0}^{\mathrm{T}} \end{pmatrix} &&= \big(I_{n-1}\,,\,\mathbf{0}\big)\, L \begin{pmatrix} I_{n-1} \\ \mathbf{0}^{\mathrm{T}} \end{pmatrix} = L_{11}. \end{aligned}$$

Since L_{11} is invertible, we can multiply both sides of the equation above by L_{11}^{-1} on the right to obtain $L_{11}X = I_{n-1}$.
□

When a non-singular matrix remains non-singular after a rank-one change, its inverse is given by the well-known Sherman-Morrison formula [19, 17]. However, when either the starting matrix or the resulting matrix after a rank-one change is singular, the pseudo-inverse is our only resort. We need the following result for a rank-one change made to a singular matrix which makes it non-singular.

Lemma 3.[24] Let A be a singular matrix, and assume $C = A + \mathbf{u}\mathbf{v}^{\mathrm{T}}$ is non-singular. Let $\mathbf{x}, \mathbf{y}$ be unit vectors (in the 2-norm) such that $A\mathbf{x} = 0$, $A^{\mathrm{T}}\mathbf{y} = 0$. Then, $\mathbf{v}^{\mathrm{T}}\mathbf{x} \neq 0$, $\mathbf{y}^{\mathrm{T}}\mathbf{u} \neq 0$, and

$$C^{-1} = A^{+} - \frac{1}{\mathbf{v}^{\mathrm{T}}\mathbf{x}}\mathbf{x}\mathbf{v}^{\mathrm{T}}A^{+} - A^{+}\frac{1}{\mathbf{y}^{\mathrm{T}}\mathbf{u}}\mathbf{u}\mathbf{y}^{\mathrm{T}} + \frac{1 + \mathbf{v}^{\mathrm{T}}A^{+}\mathbf{u}}{\mathbf{v}^{\mathrm{T}}\mathbf{x}\cdot\mathbf{y}^{\mathrm{T}}\mathbf{u}}\mathbf{x}\mathbf{y}^{\mathrm{T}} \tag{3}$$

Proof. Since C is non-singular, $C\mathbf{x} = \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{x} \neq 0$, hence $\mathbf{v}^{\mathrm{T}}\mathbf{x} \neq 0$. Suppose $\mathbf{u}$ could be written as $A\mathbf{z}$ for some $\mathbf{z}$, then $C\mathbf{z} = A\mathbf{z} + \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{z} = \mathbf{u}(1 + \mathbf{v}^{\mathrm{T}}\mathbf{z}) = C\mathbf{x}\frac{1+\mathbf{v}^{\mathrm{T}}\mathbf{z}}{\mathbf{v}^{\mathrm{T}}\mathbf{x}} \neq 0$. Hence $\mathbf{z}$ must be a multiple of $\mathbf{x}$ and $A\mathbf{z} = 0$, a contradiction. So $\mathbf{u}$ cannot be written as $A\mathbf{z}$ for any $\mathbf{z}$. Likewise $\mathbf{y}^{\mathrm{T}}\mathbf{u} \neq 0$ and $\mathbf{v}^{\mathrm{T}}$ cannot be written as $\mathbf{w}^{\mathrm{T}}A$ for any $\mathbf{w}^{\mathrm{T}}$. We thus have case (i) of [24]. Theorem 1 of [24] then yields the required result in equation (3).
□

We also need a lemma in the opposite direction, in which we apply a rank-one change to a non-singular matrix which makes it singular.

Lemma 4. Let C be an $n \times n$ non-singular matrix and suppose $A = C - \mathbf{u}\mathbf{v}^{\mathrm{T}}$ is singular. Then the Moore-Penrose pseudo-inverse of A is given as;

$$A^{+} = B \overset{\text{def}}{=} \left(I - \frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\right) C^{-1} \left(I - \frac{\mathbf{y}\mathbf{y}^{\mathrm{T}}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\right),$$

where $\mathbf{x} = C^{-1}\mathbf{u}$, $\mathbf{y}^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}C^{-1}$.

This lemma is most easily proven using the following general result.

Theorem 5 [13, Thm 3]. Let A, B be two matrices such that $\mathrm{rank}(A+B) = \mathrm{rank}(A) + \mathrm{rank}(B)$. Let $S = (P_{R(B^{\mathrm{T}})} P_{R(A^{\mathrm{T}})^{\perp}})^{+}$, $T = (P_{R(A)^{\perp}} P_{R(B)})^{+}$, where $P_{R(A)}$ denotes the orthogonal projector onto the range (column space) of a matrix A, and $P_{R(A)^{\perp}}$ denotes the orthogonal projector onto the orthogonal complement of the column space of A (same as the left nullspace of A). Then $(A+B)^{+} = (I - S)A^{+}(I - T) + SB^{+}T$.
□

Proof of Lemma 4. To prove this result, we establish some facts in sequence:

1. Let $\mathbf{z} \neq 0$ be such that $A\mathbf{z} = 0$. Then $C\mathbf{z} = \mathbf{u}\mathbf{v}^{\mathrm{T}}\mathbf{z}$. That means $C\mathbf{z}$ must be a non-zero multiple of $\mathbf{u}$. Choose the scaling such that $C\mathbf{z} = \mathbf{u}$. Then $\mathbf{z} = C^{-1}\mathbf{u} = \mathbf{x}$, $A\mathbf{x} = 0$, and $\mathbf{v}^{\mathrm{T}}\mathbf{x} = 1$. Likewise, we have $\mathbf{y}^{\mathrm{T}}A = 0$ and $\mathbf{y}^{\mathrm{T}}\mathbf{u} = 1$.

2. We have the two orthogonal projectors in the notation of Theorem 5: $\left(I - \frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\right) = P_{R(A^{\mathrm{T}})^{\perp}}$, $\left(I - \frac{\mathbf{y}\mathbf{y}^{\mathrm{T}}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\right) = P_{R(A)^{\perp}}$. Defining S and T as in Theorem 5, we then have $\left(I - \frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\right) S = 0$ and $T\left(I - \frac{\mathbf{y}\mathbf{y}^{\mathrm{T}}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\right) = 0$.

3. Hence, using (3), we get

$$\left(I - \frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\right) C^{-1} \left(I - \frac{\mathbf{y}\mathbf{y}^{\mathrm{T}}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\right) = \left(I - \frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}}\right) A^{+} \left(I - \frac{\mathbf{y}\mathbf{y}^{\mathrm{T}}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}\right) - 0 - 0 + 0 = A^{+},$$

   where we have used the fact that the left nullspace of A^{+} equals the right nullspace of A, namely $\mathrm{span}\{\mathbf{x}\}$, and likewise for the right nullspace of A^{+}.

□

3 The Laplacians

Several different Laplacians have been proposed in literature, each one helps infer different properties for graphs. We provide a brief summary here for historical perspective. Recall the notation from sec. 1. A graph $\mathcal{G}$ can be represented by its adjacency matrix A whose i, j-th entry is the weight of the edge $i \to j$, equal to one if there are no weights, or zero if there is no such edge. If $D = \text{Diag}(A \cdot \mathbf{1})$ is the diagonal matrix of row sums (out-degrees of the vertices) of A, then $P = D^{-1}A$ is the probability transition matrix for a random walk over this graph. Let $\boldsymbol{\pi}$ be the vector of stationary probabilities, such that $\boldsymbol{\pi}^{\mathrm{T}}P = \boldsymbol{\pi}^{\mathrm{T}}$ and $\boldsymbol{\pi}^{\mathrm{T}}\mathbf{1} = 1$, and let $\Pi = \text{Diag}(\boldsymbol{\pi})$. The "ordinary" **Laplacian** $L = \Pi(I - P)$ and the diagonally scaled Laplacian $L^{\mathrm{d}} = \Pi^{-1/2}L\Pi^{-1/2}$ are the main focus of this paper. We put this Laplacian in perspective by comparing it to other related Laplacians.

The **unnormalized Laplacian** $L^{\mathrm{a}} = D - A$ for an unweighted digraph yields the number of spanning trees in the graph [4].

If the underlying graph is undirected, the matrix L^{a} is also symmetric, and in fact identical (up to scaling) with L. This is because the vector of vertex degrees $A \cdot \mathbf{1}$ is a scalar multiple of $\boldsymbol{\pi}$. However, when the underlying graph is a digraph, the matrix L^{a} is not symmetric and differs from L. In [4] L^{a} is called the Formal Laplacian. This Laplacian has been used extensively to compute the average first hitting times and round-trip commute times in a random walk on an undirected graph, identifying which are the most "central" vertices [14, 16], the related effective resistance when the graph is an electrical network [5, 11, 22], including identifying minimal graph cuts in spectral graph partitioning [10, 12, 25, 26, 27, 34, 38], bounding the connectivity of the graph and related Cheeger or isoperimetric and expander constants [1, 8, 28]. The connection with electrical theory motivates the name 'Kirchoff matrix' or 'admittance matrix' for L^{a}.

The following is a classical theorem relating this Laplacian to a property of the original graph, even directed graphs, apparently first proved in [3] and later proved independently in [37], according to [4]. We present the simplest case for unweighted directed graphs. To define the terms used here, a *spanning tree rooted at a vertex k* is a subgraph of the original directed graph consisting of all the vertices and just enough directed edges so that there is exactly one path from k to any other vertex j. A *spanning arborescence rooted at k* is a subgraph consisting of all the vertices and just enough directed edges so that there is exactly one path from any vertex i back to the root k.

Theorem 6 (Matrix-Tree Theorem). Let $L^{\mathrm{a}} = D - A$ be the $n \times n$ Kirchoff matrix for an unweighted directed graph with adjacency matrix A and with $D = \text{Diag}(A \cdot \mathbf{1})$. Let $(L^{\mathrm{a}})_{-k}$ be the $(n-1) \times (n-1)$ matrix obtained from L^{a} by deleting row and column k. Then the number of spanning arborescences rooted at vertex k is equal to the principal minor $\det[(L^{\mathrm{a}})_{-k}]$.

Proof. See [4, sec. 9.6] and references therein. This is actually a special case of a more general theorem for weighted directed graphs.
□

A simple consequence of the result above is the following theorem, which holds not only when the directed graph is strongly connected but also when it has exactly one strongly connected component in the sense that all the vertices can be divided into two disjoint classes $\mathcal{V}_1, \mathcal{V}_2$ where $\mathcal{V}_1$ is strongly connected, and from each vertex in $\mathcal{V}_2$ there is a path to a vertex in $\mathcal{V}_1$.

Corollary 7. Assume the directed graph is strongly connected, or has exactly one strongly connected component. Given the notation of Theorem 6, let r_i be the number of spanning arborescences rooted at vertex i, for $i = 1, \ldots, n$. Then the vector $\mathbf{r} = (r_1, \ldots, r_n)$ is the unique (up to scaling) left annihilating vector for L^{a}.

Proof. If the graph is strongly connected, the induced Markov chain must be irreducible, and hence eigenvalue 1 of the state transition matrix P must be simple, and the stationary probabilities for the induced Markov chain must be entirely strictly positive. This implies $\mathrm{nullity}(L^{\mathrm{a}}) = 1$. By Lemma 1, its adjugate, $\mathrm{adj}(L^{\mathrm{a}})$, has rank 1. Since $L^{\mathrm{a}} \cdot \mathbf{1} = \mathbf{0}$, $\mathrm{adj}(L^{\mathrm{a}}) = \mathbf{1}\mathbf{v}^{\mathrm{T}}$ for some vector $\mathbf{v}$ unique up to scaling, which spans the left nullspace of L^{a}. By the Matrix-Tree Theorem, the diagonal entries of $\mathrm{adj}(L^{\mathrm{a}})$ are exactly the r_i's, which therefore satisfy $r_i = v_i$.
□

Directed graphs with more than one strongly connected component have no spanning arborescences, but they still have spanning forests of arborescences, extensively studied by Cheboratev et al (see [7] and references therein). The discussion of this topic is beyond the scope of this paper, but for completeness we present the following result.

Corollary 8 [7]. Using the notation of Theorem 6, the number of spanning forests of arborescences is equal to $\det(I + L^{\mathrm{a}})$.

Proof. For detailed proofs see [7]. However, a simple argument can be constructed by applying the Matrix Tree Theorem to an augmented graph obtained by adding a single new vertex to the given graph and adding an edge from every old vertex to this new vertex.
□

The **normalized Laplacian** $L^{\mathrm{p}} = I - P = D^{-1}L^{\mathrm{a}}$ has been used to analyze connectivity in terms of the mixing times or diffusion rate for the random walk as well as related expander constants, and in spectral graph partitioning. For example, in graph partitioning, L^{a} corresponds to finding the minimal cut relative to the number of vertices in each graph partition, while L^{p} corresponds to finding the minimal cut relative to the number of edges in each partition. We refer the reader to [8, 36, 38] for a detailed discussion. The **diagonally scaled Laplacian** $L^{\mathrm{d}} = \Pi^{-1/2} L \Pi^{-1/2} = I - \Pi^{1/2} P \Pi^{-1/2}$ will be studied in this paper. It is often used since in the case of undirected graphs this scaling would have the effect of symmetrizing L, hence showing that L would have all real eigenvalues.

We summarize the quantities defined above as follows, where A is the adjacency matrix for a graph $\mathcal{G}$:

$$\begin{array}{llll} D &=& \mathrm{Diag}(A \cdot \mathbf{1}) & \text{Diagonal matrix of out-degrees} \\ P &=& D^{-1}A & \text{Probability transition matrix} \\ \Pi &=& \mathrm{Diag}(\boldsymbol{\pi}) & \text{Diagonal matrix of station-} \\ && (\text{where } \boldsymbol{\pi}^{\mathrm{T}}P = \boldsymbol{\pi}^{\mathrm{T}} \text{ and } \boldsymbol{\pi}^{\mathrm{T}}\mathbf{1} = 1) & \text{ary probabilities} \\ L &=& \Pi - \Pi P & \text{Ordinary Laplacian} \\ L^{\mathrm{a}} &=& D - A = D - DP & \text{Unnormalized Laplacian} \\ L^{\mathrm{p}} &=& I - P & \text{Normalized Laplacian} \\ L^{\mathrm{d}} &=& I - \Pi^{1/2} P \Pi^{-1/2} & \text{Diagonally scaled Laplacian} \end{array} \tag{4}$$

In addition, we use the letter M to denote the Moore-Penrose pseudo-inverses of the above quantities:

$$M = L^{+}, \quad M^{\mathrm{d}} = (L^{\mathrm{d}})^{+}, \quad M^{\mathrm{p}} = (L^{\mathrm{p}})^{+}, \quad \text{etc.} \tag{5}$$

Once again, in the case of digraphs, Π is not a scalar multiple of D and L^{d} is not necessarily symmetric, unlike the situation for undirected graphs. Hence it has been found useful to study the following symmetrized Laplacians which do satisfy all the useful properties for undirected graphs.

The **symmetrized Laplacians** $L^{\mathrm{s}} = (L + L^{\mathrm{T}})/2$ **and** $L^{\mathrm{ds}} = [(L^{\mathrm{d}})^{\mathrm{T}} + L^{\mathrm{d}}]/2$ correspond to those used in [9, 39], with various diagonal scalings. In terms of the transition probability matrix (P) and the diagonal matrix of stationary probabilities (Π), we have

$$\begin{array}{lllll} L^{\mathrm{s}} &=& (L + L^{\mathrm{T}})/2 &=& \Pi - (\Pi P + P^{\mathrm{T}}\Pi)/2 \\ L^{\mathrm{ds}} &=& (L^{\mathrm{d}} + (L^{\mathrm{d}})^{\mathrm{T}})/2 &=& I - (\Pi^{1/2} P \Pi^{-1/2} + \Pi^{-1/2} P^{\mathrm{T}} \Pi^{1/2})/2 \\ &=& \Pi^{-1/2} L^{\mathrm{s}} \Pi^{-1/2} && \\ L^{\mathrm{ps}} &=& \Pi^{-1/2} L^{\mathrm{ds}} \Pi^{1/2} &=& I - (P + \Pi^{-1} P^{\mathrm{T}} \Pi)/2, \end{array} \tag{6}$$

and their corresponding pseudo-inverses

$$M^{\mathrm{s}} = (L^{\mathrm{s}})^{+}, \quad M^{\mathrm{ds}} = (L^{\mathrm{ds}})^{+}, \quad M^{\mathrm{ps}} = (L^{\mathrm{ps}})^{+}. \tag{7}$$

These Laplacians can be thought of as the ordinary Laplacians for a weighted undirected graph $\mathcal{G}^{\mathrm{s}}$ derived from the original directed graph $\mathcal{G}$. Assume $\mathcal{G}$ is a directed graph without self-loops (edges starting and ending on the same vertex), with adjacency matrix A. The derived weighted undirected graph $\mathcal{G}^{\mathrm{s}}$ is defined to be the graph with adjacency matrix $A^{\mathrm{s}} = (\Pi P + P^{\mathrm{T}}\Pi)/2$. The new graph $\mathcal{G}^{\mathrm{s}}$ has exactly the same vertices as $\mathcal{G}$ and has edges between a pair of vertices exactly where there is an edge in either direction in $\mathcal{G}$. The weight on the edge in $\mathcal{G}^{\mathrm{s}}$ connecting vertices i and j is

$$a^{\mathrm{s}}_{ij} = a^{\mathrm{s}}_{ji} = \frac{\pi_i a_{ij}}{2d_i} + \frac{\pi_j a_{ji}}{2d_j} = 1/2(\pi_i p_{ij} + \pi_j p_{ji}), \tag{8}$$

where a_{ij} is the weight of the edge $i \to j$ in the original graph $\mathcal{G}$, equal to one if $\mathcal{G}$ was unweighted, and d_i is the [weighted] out-degree of node i in $\mathcal{G}$. The new matrix of transition probabilities is

$$P^{\mathrm{s}} = \Pi^{-1} A^{\mathrm{s}} = (P + \Pi^{-1} P^{\mathrm{T}} \Pi)/2 \tag{9}$$

with individual entries

$$p^{\mathrm{s}}_{ij} = \frac{1}{2} \cdot \left(p_{ij} + \frac{\pi_j p_{ji}}{\pi_i} \right) = \frac{\pi_i p_{ij} + \pi_j p_{ji}}{2\pi_i}. \tag{10}$$

The stationary probabilities for Markov chain represented by P^{s} match those for P: $\boldsymbol{\pi}^{\mathrm{T}} P^{\mathrm{s}} = \boldsymbol{\pi}^{\mathrm{T}}$. A simple calculation shows that the symmetrized Laplacians, originally defined by symmetrizing the Laplacians of $\mathcal{G}$, are also the usual Laplacians corresponding to the weighted undirected graph $\mathcal{G}^{\mathrm{s}}$:

$$\begin{aligned} L^{\mathrm{s}} &= \Pi - \Pi P^{\mathrm{s}} \\ L^{\mathrm{ds}} &= \Pi^{-1/2} L^{\mathrm{s}} \Pi^{-1/2} = I - \Pi^{1/2} P^{\mathrm{s}} \Pi^{-1/2} \\ L^{\mathrm{ps}} &= \Pi^{-1} L^{\mathrm{s}} = I - P^{\mathrm{s}} \end{aligned} \tag{11}$$

This construction shows that the bounds for $\mathcal{G}$ in [9, 39] can be treated as bounds for the undirected graph $\mathcal{G}^{\mathrm{s}}$ based on the classical theory for undirected graphs. In [9], they use both L^{ds} and L^{s}, referring to L^{s} as the "combinatorial Laplacian" and reserving the name of just "Laplacian" to our diagonally scaled version L^{ds}. In [39], they use only L^{ds}. In Sec. 5 we derive bounds on the commute times in terms of the stationary probabilities, which also apply to $\mathcal{G}^{\mathrm{s}}$, limiting how much the commute times for $\mathcal{G}^{\mathrm{s}}$ graph can differ from those of the original $\mathcal{G}$. We remark that an alternative way to symmetrize a directed graph $\mathcal{G}$, with an asymmetric adjacency matrix A, is to symmetrize the edges to create $A^{\mathrm{v}} = (A + A^{\mathrm{T}})/2$ [6]. We denote this naively symmetrized graph by $\mathcal{G}^{\mathrm{v}}$. In latter sections we make empirical comparisons for random walk measures to reveal the varying degrees of inaccuracy incurred upon approximating a directed graph $\mathcal{G}$ by either $\mathcal{G}^{\mathrm{s}}$ or $\mathcal{G}^{\mathrm{v}}$.

Henceforth, we concentrate on the asymmetric Laplacian $L = \Pi(I - P)$, referring to this as simply the "Laplacian," as well as the diagonally scaled Laplacian $L^{\mathrm{d}} = \Pi^{1/2}(I - P)\Pi^{-1/2}$. We derive bounds applicable to the directed graph itself based on these Laplacians, separate from bounds for the related undirected graph.

4 Fundamental Matrix

Consider a Markov chain with state transition matrix $P = D^{-1}A$, where $D = \mathrm{Diag}(\mathbf{d})$ is a diagonal matrix, $\mathbf{d} = A \cdot \mathbf{1}$ is the vector of [weighted] out-degrees for the vertices of the graph, and $\mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}$. In the following, we assume the graph is directed and strongly connected, or equivalently the Markov chain is irreducible and has no transient states. Clearly, we do not assume either reversibility or aperiodicity of the equivalent Markov chain.

Definition 9. Using the quantities defined in (4), we define the Fundamental matrix for a digraph or its corresponding Markov chain, under various scalings:

(a) The *Fundamental Matrix* Z^{p} [18] whose inverse is

$$(Z^{\mathrm{p}})^{-1} \stackrel{\mathrm{def}}{=} Y^{\mathrm{p}} = (L^{\mathrm{p}} + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}}) = (I - P + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}}). \tag{12}$$

(b) The *scaled Fundamental Matrix*, $\widetilde{Z} = Z^{\mathrm{p}}\Pi^{-1}$ whose inverse is

$$\widetilde{Z}^{-1} \stackrel{\text{def}}{=} \widetilde{Y} = \Pi Y^{\mathrm{p}} = L + \boldsymbol{\pi}\boldsymbol{\pi}^{\mathrm{T}} = \Pi(I - P + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}}). \tag{13}$$

(c) The *diagonally scaled Fundamental Matrix* $Z^{\mathrm{d}} = \Pi^{1/2}\widetilde{Z}\Pi^{1/2} = \Pi^{-1/2}Z^{\mathrm{p}}\Pi^{1/2}$ whose inverse is

$$(Z^{\mathrm{d}})^{-1} \stackrel{\text{def}}{=} Y^{\mathrm{d}} = \Pi^{1/2}Y^{\mathrm{p}}\Pi^{-1/2} = L^{\mathrm{d}} + \sqrt{\boldsymbol{\pi}}\sqrt{\boldsymbol{\pi}}^{\mathrm{T}}. \tag{14}$$

Here we use the shorthand $\sqrt{\boldsymbol{\pi}} = (\sqrt{\pi_1}, \ldots, \sqrt{\pi_n})^{\mathrm{T}}$ for the vector obtained by taking the square root of each element. We remark that this vector is a unit vector in 2-norm since $\|\sqrt{\boldsymbol{\pi}}\|_2^2 = \sum_i \pi_i = 1$.

Lemma 10. The following are some of the elementary properties of the laplacians, their respective pseudo-inverses, and the inverses of the fundamental matrices, under various scalings,

$$
\begin{aligned}
L \cdot \mathbf{1} &= L^{\mathrm{T}}\mathbf{1} = \mathbf{0} && (15)\\
M \cdot \mathbf{1} &= M^{\mathrm{T}} \cdot \mathbf{1} = \mathbf{0} && (16)\\
L^{\mathrm{d}} \cdot \sqrt{\boldsymbol{\pi}} &= (L^{\mathrm{d}})^{\mathrm{T}} \cdot \sqrt{\boldsymbol{\pi}} = \mathbf{0} && (17)\\
M^{\mathrm{d}} \cdot \sqrt{\boldsymbol{\pi}} &= (M^{\mathrm{d}})^{\mathrm{T}} \cdot \sqrt{\boldsymbol{\pi}} = \mathbf{0} && (18)\\
&L,\ L^{\mathrm{d}},\ M,\ M^{\mathrm{d}} \text{ all have (left and right) nullity equal to 1} && (19)\\
\widetilde{Y} \cdot \mathbf{1} &= \widetilde{Y}^{\mathrm{T}} \cdot \mathbf{1} = \boldsymbol{\pi} && (20)\\
Y^{\mathrm{d}} \cdot \sqrt{\boldsymbol{\pi}} &= (Y^{\mathrm{d}})^{\mathrm{T}} \cdot \sqrt{\boldsymbol{\pi}} = \sqrt{\boldsymbol{\pi}} && (21)
\end{aligned}
$$

Proof. To prove (15) and (19), observe $\mathbf{0} = L\mathbf{x} = \Pi\mathbf{x} - \Pi P\mathbf{x} \iff \mathbf{x} = P\mathbf{x}$. For a strongly connected Markov chain, the Perron-Frobenius theory [15, 19] guarantees that the eigenvalue 1 of P is simple and hence $\mathbf{x}$ must be a multiple of $\mathbf{1}$. (16) follows from the observation that the right annihilating vector for L is the left annihilating vector for M and viceversa. The rest follows similarly.
□

Before we go any further, we must first establish that $\widetilde{Z}$ indeed exists, or equivalently $\widetilde{Y}$ is invertible [18]. We can actually prove the following stronger result.

Theorem 11. Let P be the transition matrix for an irreducible Markov chain with a vector of stationary probabilities $\boldsymbol{\pi}$, and $\Pi = \mathrm{Diag}(\boldsymbol{\pi})$. Then $\widetilde{Y} = \Pi(I - P) + \boldsymbol{\pi}\boldsymbol{\pi}^{\mathrm{T}}$ is non-singular and is also positive definite (in the sense that $Y^{\mathrm{s}} = (\widetilde{Y} + \widetilde{Y}^{\mathrm{T}})/2$ is symmetric positive definite in the usual sense). In addition L is positive semi-definite in the sense that $L^{\mathrm{s}} = (L + L^{\mathrm{T}})/2$ is positive semi-definite.

The following lemma is useful in proving Theorem 11.

Lemma 12. For any given real matrix A, its "symmetric part", $(A + A^{\mathrm{T}})/2$, is symmetric positive semi-definite if and only if $\mathbf{x}^{\mathrm{T}}A\mathbf{x} \geq 0$ for any real vector $\mathbf{x}$. We say the *real field of values* for A is non-negative.

Proof of Lemma 12. The symmetry of $A+A^{\mathrm{T}}$ is trivial. Within this proof, let $i=\sqrt{-1}$ and let $\square^{\mathrm{H}}$ denote the conjugate transpose of $\square$. ("if") Let $\mathbf{z}=\mathbf{x}+i\mathbf{y}$ for any real vectors $\mathbf{x},\mathbf{y}$. Then $\mathbf{z}^{\mathrm{H}}A\mathbf{z}=(\mathbf{x}^{\mathrm{T}}-i\mathbf{y}^{\mathrm{T}})A(\mathbf{x}+i\mathbf{y})=\mathbf{x}^{\mathrm{T}}A\mathbf{x}+\mathbf{y}^{\mathrm{T}}A\mathbf{y}+i(\mathbf{x}^{\mathrm{T}}A\mathbf{y}-\mathbf{y}^{\mathrm{T}}A\mathbf{x})=\alpha+i\beta$, where $\alpha\geq 0$. Hence $\mathbf{z}^{\mathrm{H}}(A+A^{\mathrm{T}})\mathbf{z}=2\alpha\geq 0$. ("only if") Suppose $A+A^{\mathrm{T}}$ is real symmetric positive semi-definite. Then for any real vector $\mathbf{x}$: $\mathbf{x}^{\mathrm{T}}A\mathbf{x}=\mathbf{x}^{\mathrm{H}}(A+A^{\mathrm{T}})\mathbf{x}/2\geq 0$.
□

In light of Lemma 12, we say a general real matrix A is positive semi-definite if and only if $A+A^{\mathrm{T}}$ is symmetric semi-positive definite in the usual sense.

Proof of Theorem 11. Let $L=\widetilde{Y}-\boldsymbol{\pi}\boldsymbol{\pi}^{\mathrm{T}}$, and $A=I-(L+L^{\mathrm{T}})/2\stackrel{\mathrm{def}}{=}I-L^{\mathrm{s}}$. From (15) notice that $A\mathbf{1}=A^{\mathrm{T}}\mathbf{1}=\mathbf{1}$. A is symmetric and doubly stochastic with non-negative entries. Actually, all the entries are in the interval $[0,1)$. It is also irreducible, since the original P was, so 1 is a simple eigenvalue of A, and all the other eigenvalues are in the interval $[-1,+1)$. Therefore, $L^{\mathrm{s}}=(L+L^{\mathrm{T}})/2$ has a simple zero eigenvalue and all its other eigenvalues are in the interval $(0,2]$. Hence L^{s} is positive semi-definite and nullity$(L^{\mathrm{s}})=1$. This implies that the 'real field of values' for L is non-negative: $\mathbf{x}^{\mathrm{T}}L\mathbf{x}\geq 0$ for any real $\mathbf{x}$.

We further observe that $\mathbf{x}^{\mathrm{T}}L\mathbf{x}=0$ only when $\mathbf{x}=\alpha\mathbf{1}$ for some scalar α. Observe that $\mathbf{x}^{\mathrm{T}}\widetilde{Y}\mathbf{x}=\mathbf{x}^{\mathrm{T}}L\mathbf{x}+(\mathbf{x}^{\mathrm{T}}\boldsymbol{\pi})(\boldsymbol{\pi}^{\mathrm{T}}\mathbf{x})\geq 0$, with $\mathbf{x}^{\mathrm{T}}L\mathbf{x}\geq 0$ and $(\mathbf{x}^{\mathrm{T}}\boldsymbol{\pi})(\boldsymbol{\pi}^{\mathrm{T}}\mathbf{x})\geq 0$. The only vector $\mathbf{x}$ for which both $\mathbf{x}^{\mathrm{T}}L\mathbf{x}=0$ and $(\mathbf{x}^{\mathrm{T}}\boldsymbol{\pi})(\boldsymbol{\pi}^{\mathrm{T}}\mathbf{x})=0$ is $\mathbf{x}=0$. Hence $\mathbf{x}^{\mathrm{T}}\widetilde{Y}\mathbf{x}>0$ for any real $\mathbf{x}\neq\mathbf{0}$.
□

As an application, the Fundamental Matrix can be used to compute the "Hitting Time", also known in the literature as the "First Transit Time" or "First Passage Time" in a random walk over the underlying digraph. Let $\mathtt{H}(i,j)$ be the average number of state transitions required to reach state j for the first time starting from state i (hitting time). Similarly, let $\mathtt{C}(i,j)$ be the average "Commute Time" defined as the average number of steps taken in a random walk starting from state i, visit state j for the first time, and return back to state i. Evidently, $\mathtt{C}(i,j)=\mathtt{H}(i,j)+\mathtt{H}(j,i)$.

Theorem 13. Define the Fundamental Matrices according to Def. 9. Then the one-way expected hitting times are

$$\mathtt{H}(i,j)=\frac{z^{\mathrm{p}}_{jj}-z^{\mathrm{p}}_{ij}}{\pi_j}=\tilde{z}_{jj}-\tilde{z}_{ij}=\frac{z^{\mathrm{d}}_{jj}}{\pi_j}-\frac{z^{\mathrm{d}}_{ij}}{\sqrt{\pi_i\pi_j}} \tag{22}$$

The round-trip expected commute times are then

$$\mathtt{C}(i,j)=\frac{z^{\mathrm{p}}_{jj}-z^{\mathrm{p}}_{ij}}{\pi_j}+\frac{z^{\mathrm{p}}_{ii}-z^{\mathrm{p}}_{ji}}{\pi_i}=\tilde{z}_{ii}+\tilde{z}_{jj}-\tilde{z}_{ij}-\tilde{z}_{ji}=\frac{z^{\mathrm{d}}_{jj}}{\pi_j}+\frac{z^{\mathrm{d}}_{ii}}{\pi_i}-\frac{z^{\mathrm{d}}_{ij}+z^{\mathrm{d}}_{ji}}{\sqrt{\pi_i\pi_j}} \tag{23}$$

In matrix form,

$$\begin{aligned}\mathtt{H}&=\mathbf{1}\cdot[\mathrm{diag}(\widetilde{Z})]^{\mathrm{T}}-\widetilde{Z}\\ \mathtt{C}&=\mathtt{H}+\mathtt{H}^{\mathrm{T}}=\mathbf{1}\cdot[\mathrm{diag}(\widetilde{Z})]^{\mathrm{T}}+[\mathrm{diag}(\widetilde{Z})]\cdot\mathbf{1}^{\mathrm{T}}-\widetilde{Z}-\widetilde{Z}^{\mathrm{T}}\end{aligned} \tag{24}$$

Proof. The first part of formula (22) is proved in [18] starting with the recursive formula for $\mathtt{H}(i,j)$ [21, 29, 31, 18]:

$$\mathtt{H}(i,j) = 1 + \sum_{\ell=1}^{n} p_{i\ell}\mathtt{H}(\ell,j), \qquad \text{for} \qquad i = 1,\ldots,n, \tag{25}$$

where by convention, $\mathtt{H}(\ell,\ell) = 0$, $\forall \ell$. The last parts of (22) and (24) were shown in [23] and follow from the identity $\widetilde{Z} = \Pi^{-1/2} Z^{\rm d} \Pi^{-1/2}$. The rest follows by direct calculation or by simply assembling the scalar formulas into a matrix formulation. Notice that $\mathtt{C}(i,j) = \mathtt{H}(i,j) + \mathtt{H}(j,i)$ is a symmetric quantity while $\mathtt{H}(i,j)$ is generally not, whether the underlying graph is directed or undirected.
□

The following lemma relates the pseudo-inverse of the Laplacians to the Fundamental Matrix.

Lemma 14. Using the notation of Def. 9, for a strongly connected directed graph,

(a) $M = L^{+} = \left(I - \frac{\mathbf{1}\mathbf{1}^{\rm T}}{n}\right) \widetilde{Z} \left(I - \frac{\mathbf{1}\mathbf{1}^{\rm T}}{n}\right)$.

(b) $\widetilde{Z} = M - M\boldsymbol{\pi}\mathbf{1}^{\rm T} - \mathbf{1}\boldsymbol{\pi}^{\rm T} M + (1 + \boldsymbol{\pi}^{\rm T} M \boldsymbol{\pi})\mathbf{1}\mathbf{1}^{\rm T}$.

(c) $\tilde{z}_{ij} = m_{ij} - \sum_j m_{ij}\pi_j - \sum_i m_{ij}\pi_i + (1 + \sum_{ij} m_{ij}\pi_i\pi_j)$.

(d) L, $M{=}L^{+}$, $L^{\rm d}$, $M^{\rm d}{=}(L^{\rm d})^{+}$ are all positive semi-definite.

(e) $Z^{\rm d} = M^{\rm d} + \sqrt{\boldsymbol{\pi}}\sqrt{\boldsymbol{\pi}}^{\rm T}$.

Proof. Noting that $L\cdot\mathbf{1} = 0$, $L^{\rm T}\cdot\mathbf{1} = 0$, $\widetilde{Y}\cdot\mathbf{1} = \boldsymbol{\pi}$, $\widetilde{Y}^{\rm T}\cdot\mathbf{1} = \boldsymbol{\pi}$ (Lemma 10), formulas (a), (b) follow immediately from Lemmas 4 and 3, respectively. Formula (c) is just the elementwise version of item (b). Formula (d) follows from Theorem 11 and Lemma 4. Formula (e) follows similarly to item (b) by recalling (18).
□

Theorem 15. If L is the Laplacian for a strongly connected unweighted directed graph, and $M = L^{+}$ is its Moore-Penrose pseudo-inverse, then the expected hitting times and commute times, in terms of the Laplacian pseudo-inverse, are

$$\mathtt{H}(i,j) = m_{jj} - m_{ij} + \sum_{\ell}(m_{i\ell} - m_{j\ell})\pi_\ell = \frac{m^{\rm d}_{jj}}{\pi_j} - \frac{m^{\rm d}_{ij}}{\sqrt{\pi_i\pi_j}} \tag{26}$$

$$\mathtt{C}(i,j) = m_{jj} + m_{ii} - m_{ij} - m_{ji} = \frac{m^{\rm d}_{jj}}{\pi_j} + \frac{m^{\rm d}_{ii}}{\pi_i} - \frac{m^{\rm d}_{ij} + m^{\rm d}_{ji}}{\sqrt{\pi_i\pi_j}} \tag{27}$$

Furthermore, a set of points $\{\mathbf{s}_i\}_1^n$ can be found in the Euclidean space $\mathbb{R}^n$ corresponding to the n vertices of the graph such that $\mathtt{C}(i,j) = \|\mathbf{s}_i - \mathbf{s}_j\|_2^2$.

Proof. Substitute Lemma 14(c), (e) into the formulas of Theorem 13. The relations involving m^{d}_{ij} were shown in [23]. The last statement follows by observing that $M+M^{\mathrm{T}}$ is positive semi-definite or equivalently that $\widetilde{Z}+\widetilde{Z}^{\mathrm{T}}$ is positive definite, so that they can be considered as Gram matrices. It is then a simple consequence of (23) and the following theorem of [32, 33, 2], reformulated in terms of matrices.

Theorem 16.[32, 33, 2] Let $\mathbf{Z}$ be an $n \times n$ symmetric matrix. Define the matrix $\mathtt{C} = [\mathrm{diag}(\mathbf{Z}) \cdot \mathbf{1}^{\mathrm{T}} + \mathbf{1} \cdot \mathrm{diag}(\mathbf{Z})^{\mathrm{T}}]/2 - \mathbf{Z}$. Then there exists a set of points $\{\mathbf{s}_i\}_1^n \subset \mathbb{R}^n$ such that $\mathtt{C}_{ij} = \|\mathbf{s}_i - \mathbf{s}_j\|_2^2 \; \forall i, j = 1, \ldots, n$ if and only if $\mathbf{Z}$ is positive semi-definite.
□

The formulas of Theorem 15 reduce to the usual known formulas for hitting times and commute times when the underlying graph is undirected [5, 11, 14, 16, 22].

5 Bounds on Commute Times

In this section we give some upper and lower bounds on the commute times in terms of the transition probabilities and the stationary probabilities. First we recall the following fact:

$$1 + \sum_i p_{ki} \mathtt{H}(i,k) = \frac{1}{\pi_k} \tag{28}$$

The left side of (28) is the expected return time between visits to node k in the Markov chain modeled by transition matrix P, computed by taking the weighted average of the hitting times from each of k's neighbors back to k. It is well known that this is equal to the inverse of the stationary probability. A purely linear algebraic derivation of this fact is as follows. First observe

$$\begin{aligned} -P\widetilde{Z} &= (I - P + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}})\widetilde{Z} - (I + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}})\widetilde{Z} = Y^{\mathrm{P}}\widetilde{Z} - (I + \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}})\widetilde{Z} \\ &= \Pi^{-1} - \widetilde{Z} - \mathbf{1}\boldsymbol{\pi}^{\mathrm{T}}\widetilde{Z} = \Pi^{-1} - \widetilde{Z} - \mathbf{1}\mathbf{1}^{\mathrm{T}}, \end{aligned}$$

where the last equality uses (20). Next combine the above with (24) to obtain

$$\begin{aligned} P\,\mathtt{H} &= P(\mathbf{1} \cdot [\mathrm{diag}(\widetilde{Z})]^{\mathrm{T}} - \widetilde{Z}) \\ &= \mathbf{1} \cdot [\mathrm{diag}(\widetilde{Z})]^{\mathrm{T}} + \Pi^{-1} - \widetilde{Z} - \mathbf{1}\mathbf{1}^{\mathrm{T}} = \mathtt{H} + \Pi^{-1} - \mathbf{1}\mathbf{1}^{\mathrm{T}}. \end{aligned}$$

Equating the diagonal entries and observing that $\mathtt{H}(k,k) = 0$ for all k yields formula (28).

Since the average round-trip commute time between node k and some other specific node j must be at least equal to the average time from k to any other node and back to node k, we immediately have a lower bound

$$\mathtt{C}(i,j) = \mathtt{C}(j,i) \geq \max\left\{\frac{1}{\pi_i}, \frac{1}{\pi_j}\right\}. \tag{29}$$

We also have the following identity that follows immediately from (22)

$$\mathtt{H}\boldsymbol{\pi} = \mathbf{1}\cdot \operatorname{diag}(\widetilde{Z})^{\mathrm{T}}\boldsymbol{\pi} - \widetilde{Z}\boldsymbol{\pi} = (\operatorname{tr}(Z^{\mathrm{p}})-1)\mathbf{1} \quad\Longleftrightarrow\quad \sum_k \mathtt{H}(i,k)\pi_k = \operatorname{tr}(Z^{\mathrm{p}})-1, \tag{30}$$

where $\operatorname{tr}(Z)$ denotes the *trace* of the matrix Z (the sum of the diagonal entries). By observing that all the factors in (30) are non-negative, we also have the following upper bounds.

$$\begin{aligned} \mathtt{H}(i,j) &\le (\operatorname{tr}(Z^{\mathrm{p}})-1)/\pi_j. \\ \mathtt{C}(i,j) &\le (\operatorname{tr}(Z^{\mathrm{p}})-1)\cdot\left(\frac{1}{\pi_j}+\frac{1}{\pi_i}\right). \end{aligned} \tag{31}$$

Observe that $\operatorname{tr}(Z^{\mathrm{p}}) = \operatorname{tr}(Z^{\mathrm{d}})$, and this last quantity can be written in terms of $\boldsymbol{\pi}$ and the diagonally scaled Laplacian L^{d} as follows. Recalling (14), we construct the symmetric unitary Householder transformation $H = I - 2\frac{\mathbf{v}\mathbf{v}^{\mathrm{T}}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}$ such that $H\sqrt{\boldsymbol{\pi}} = \mathbf{e}_1$, by setting $\mathbf{v} = \sqrt{\boldsymbol{\pi}} - \mathbf{e}_1$. By (18), (21), $HL^{\mathrm{d}}H$, $HY^{\mathrm{d}}H$ have the form:

$$L^{\mathrm{h}} = HL^{\mathrm{d}}H = \begin{pmatrix} 0 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & L_2^{\mathrm{h}} \end{pmatrix} \quad\text{and}\quad Y^{\mathrm{h}} = HY^{\mathrm{d}}H = \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & L_2^{\mathrm{h}} \end{pmatrix},$$

yielding the identity $(Y^{\mathrm{h}})^{-1} = (L^{\mathrm{h}})^{+} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}$, equivalent to (14). Hence $\operatorname{Tr}[(Y^{\mathrm{d}})^{-1}] = \operatorname{Tr}[(L^{\mathrm{d}})^{+}] + 1$. This immediately yields the identity

$$\operatorname{tr}(Z^{\mathrm{p}}) = \operatorname{tr}(Z^{\mathrm{d}}) = \operatorname{tr}[M^{\mathrm{d}}] + 1. \tag{32}$$

For the corresponding weighted undirected graph $\mathcal{G}^{\mathrm{s}}$ represented by (8) sharing the same stationary probabilities as $\mathcal{G}$, both the lower bound (29) and the upper bound (31) apply unchanged, though the factor $\operatorname{tr}(Z^{\mathrm{p}}) - 1 = \operatorname{tr}[M^{\mathrm{d}}]$ in the upper bound will be replaced by $\operatorname{tr}(Z^{\mathrm{ps}}) - 1 = \operatorname{tr}[(L^{\mathrm{ds}})^{+}] = \operatorname{tr}[M^{\mathrm{ds}}]$. We now show that the resulting upper bound applies not only to $\mathcal{G}^{\mathrm{s}}$, but also to the original $\mathcal{G}$, so that we have a set of upper and lower bounds common to both graphs. These bounds will imply that there is a limit to how much difference there can be between the commute times for a directed graph $\mathcal{G}$ and those for its corresponding symmetrized graph $\mathcal{G}^{\mathrm{s}}$. To show this, we need the following lemmas.

Lemma 17. If $\mathbf{A}$ is real symmetric positive definite and $\mathbf{B}$ is real skew-symmetric ($\mathbf{B}^{\mathrm{T}} = -\mathbf{B}$), then $\mathbf{C} = \mathbf{A}^{-1} - (\mathbf{A}+\mathbf{B})^{-1}$ exists and is real positive semi-definite (in the sense that $\mathbf{u}^{\mathrm{T}}\mathbf{C}\mathbf{u} \ge 0$ for any real $\mathbf{u}$). If $\mathbf{B}$ is also non-singular, then $\mathbf{C}$ is positive definite.

Proof.

1. Recall skew-symmetry implies $\mathbf{u}^{\mathrm{T}}\mathbf{B}\mathbf{u} = 0$ for any real vector $\mathbf{u}$.

2. Check $\mathbf{C}$ exists: For any nonzero vector $\mathbf{u}$, $\mathbf{u}^{\mathrm{T}}(\mathbf{A}+\mathbf{B})\mathbf{u} = \mathbf{u}^{\mathrm{T}}\mathbf{A}\mathbf{u} > 0$. So $(\mathbf{A}+\mathbf{B})$ cannot be a singular matrix.

3. For a non-zero vector $\mathbf{u}$, set $\mathbf{v} = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{u}$, and notice $\mathbf{v}^{\mathrm{T}} = \mathbf{u}^{\mathrm{T}}(\mathbf{A}-\mathbf{B})^{-1}$ due to the skew-symmetry of $\mathbf{B}$. Compute

$$\mathbf{u}^{\mathrm{T}}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{u} = \mathbf{v}^{\mathrm{T}}(\mathbf{A}-\mathbf{B})\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{A}\mathbf{v}.$$

4. Compute

$$\begin{aligned}\mathbf{u}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u} &= \mathbf{v}^{\mathrm{T}}(\mathbf{A}-\mathbf{B})\mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}}(\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{B}\mathbf{A}^{-1}\mathbf{B})\mathbf{v} \\ &= \mathbf{v}^{\mathrm{T}}\mathbf{A}\mathbf{v}-\mathbf{v}^{\mathrm{T}}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{v}.\end{aligned}$$

5. Now $-\mathbf{B}\mathbf{A}^{-1}\mathbf{B} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B}$ is a symmetric positive semi-definite matrix (strictly definite if $\mathbf{B}$ non-singular). Hence we have, for any non-zero vector $\mathbf{u}$,

$$\begin{aligned}\mathbf{u}^{\mathrm{T}}\mathbf{C}\mathbf{u} &= \mathbf{u}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{u}-\mathbf{u}^{\mathrm{T}}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{u} \\ &= -\mathbf{v}^{\mathrm{T}}\mathbf{B}\mathbf{A}^{-1}\mathbf{B}\mathbf{v} \\ &\geq 0 \text{ (strictly} > 0 \text{ if } \mathbf{B} \text{ is non-singular).}\end{aligned}$$

□

Lemma 18. If Y is a real matrix and $(Y+Y^{\mathrm{T}})$ is positive definite, then $\mathrm{tr}(Y^{-1}) \leq \mathrm{tr}[((Y+Y^{\mathrm{T}})/2)^{-1}]$.

Proof.

Let $\mathbf{A} = (Y+Y^{\mathrm{T}})/2$. This matrix is positive definite. Let $\mathbf{B} = (Y-Y^{\mathrm{T}})/2$. This matrix is real skew-symmetric. Let $\mathbf{C} = \mathbf{A}^{-1}-Y^{-1} = \mathbf{A}^{-1}-(\mathbf{A}+\mathbf{B})^{-1}$. Then $\mathbf{C}$ is positive definite and $\mathrm{tr}(\mathbf{A}^{-1})-\mathrm{tr}(Y^{-1}) = \mathrm{tr}(\mathbf{C}) \geq 0$.

□

Lemma 19. Let $\mathcal{G}$ be a directed graph with probability transition matrix P and let $\mathcal{G}^{\mathrm{s}}$ be the corresponding weighted undirected graph with the associated matrices defined by (4), (6), and Def. 9. Then $\mathrm{tr}(Z^{\mathrm{d}}) \leq \mathrm{tr}(Z^{\mathrm{ps}}) = \mathrm{tr}[M^{\mathrm{ds}}]+1$, and $\mathtt{C}(i,j) < \mathtt{C}^{\mathrm{s}}(i,j)$.

Proof. Since $Y^{\mathrm{d}} = \Pi^{-1/2}\widetilde{Y}\Pi^{-1/2} = \Pi^{-1/2}\widetilde{Z}^{-1}\Pi^{-1/2}$ (a nonsingular congruence transformation), it follows by Theorem 11 that its symmetric part, $Y^{\mathrm{ds}} = 1/2(Y^{\mathrm{d}}+(Y^{\mathrm{d}})^{\mathrm{T}})$, is positive definite. Hence Lemma 18 applies guaranteeing that $\mathrm{tr}(Z^{\mathrm{d}}) \leq \mathrm{tr}(Z^{\mathrm{ds}})$. In addition, defining $Y^{\mathrm{s}} = (Z^{\mathrm{s}})^{-1} = L^{\mathrm{s}}+\boldsymbol{\pi}\boldsymbol{\pi}^{\mathrm{T}} = (L+L^{\mathrm{T}})/2+\boldsymbol{\pi}\boldsymbol{\pi}^{\mathrm{T}}$, Lemma 18 guarantees that $X = Z^{\mathrm{s}}-\widetilde{Z}$ is also positive semi-definite. Combining (23) with $\mathtt{C}^{\mathrm{s}}(i,j) = z^{\mathrm{s}}_{ii}+z^{\mathrm{s}}_{jj}-z^{\mathrm{s}}_{ij}-z^{\mathrm{s}}_{ji}$ and Theorem 16 yields the fact that $\partial\mathtt{C}(i,j) = \mathtt{C}^{\mathrm{s}}(i,j)-\mathtt{C}(i,j) = x_{ii}+x_{jj}-x_{ij}-x_{ji}$ is also a squared euclidean distance and hence non-negative.

□

Theorem 20. Let $\mathcal{G}$, $\mathcal{G}^{\mathrm{s}}$, P, P^{s}, L^{d}, L^{ds} be defined as in Lemma 19. Then the respective expected hitting $\mathtt{H},\mathtt{H}^{\mathrm{s}}$ and commute times $\mathtt{C},\mathtt{C}^{\mathrm{s}}$ satisfy the following bounds

$$\begin{array}{llll}\text{(a) } \mathtt{H}(i,j) & \leq \mathrm{tr}[M^{\mathrm{d}}]/\pi_j & \leq & \mathrm{tr}[M^{\mathrm{ds}}]/\pi_j; \\ \text{(b) } \mathtt{H}^{\mathrm{s}}(i,j) & & \leq & \mathrm{tr}[M^{\mathrm{ds}}]/\pi_j;\end{array} \tag{33}$$

$$\begin{array}{ll}\text{(a) } \max\left\{\frac{1}{\pi_i},\frac{1}{\pi_j}\right\} \leq \mathtt{C}(i,j) \leq \mathrm{tr}[M^{\mathrm{d}}]\cdot\left(\frac{1}{\pi_j}+\frac{1}{\pi_i}\right) & \leq \mathrm{tr}[M^{\mathrm{ds}}]\cdot\left(\frac{1}{\pi_j}+\frac{1}{\pi_i}\right) \\ \text{(b) } \max\left\{\frac{1}{\pi_i},\frac{1}{\pi_j}\right\} \leq \mathtt{C}(i,j) \leq \qquad \mathtt{C}^{\mathrm{s}}(i,j) & \leq \mathrm{tr}[M^{\mathrm{ds}}]\cdot\left(\frac{1}{\pi_j}+\frac{1}{\pi_i}\right);\end{array} \tag{34}$$

Proof. Follows from the above discussion.

□

We remark that all the eigenvalues of L^{ds} are real and positive (except for one zero eigenvalue) and are identical to the eigenvalues of L^{ps}, since these are the appropriately scaled Laplacians for an undirected graph. If we enumerate these eigenvalues in non-decreasing order, $0 < \lambda_2 \leq \cdots \leq \lambda_n$, then an upper bound for the factor $\mathrm{tr}[M^{\mathrm{ds}}]$ is $\mathrm{tr}[M^{\mathrm{ds}}] \leq (n-1)/\lambda_2$. This theorem is one example in which quantities derived from an undirected graph, for which much theory is known, can be applied to bound a property for a strongly connected directed graph.

6 Estimating Centrality of Individual Nodes

As a possible application, we can get a measure of the centrality of a given vertex by adding the average lengths of all walks between any pair of vertices when those walks are restricted to passing through the given vertex, following similar analysis for undirected graphs [30]. If we compare this sum to the sum over all possible paths, we get an estimate on how much the restriction of passing through a given vertex q represents a detour in going from an arbitrary vertex i to another arbitrary vertex j. Since $\sum_i m_{ij} = \sum_j m_{ij} = 0$ (16), equation (26) yields (for all paths)

$$\sum_{ij} \mathtt{H}(i,j) = n \sum_{j} m_{jj} = n \cdot \mathrm{Trace}(M). \tag{35}$$

The expected length of a walk from i to j forced through node q is:

$$\begin{aligned} \mathtt{H}_q(i,j) &= \mathtt{H}(i,q) + \mathtt{H}(q,j) \\ &= m_{qq} - m_{iq} + \sum_{\ell} (m_{i\ell} - m_{q\ell})\pi_\ell \\ &\quad + m_{jj} - m_{qj} + \sum_{\ell} (m_{q\ell} - m_{j\ell})\pi_\ell \\ &= m_{qq} + m_{jj} - m_{iq} - m_{qj} + \sum_{\ell} (m_{i\ell} - m_{j\ell})\pi_\ell \\ &= \mathtt{H}(i,j) + m_{qq} - m_{iq} - m_{qj} \end{aligned} \tag{36}$$

Summing this up for all pairs of sources i and destinations j yields

$$\sum_{ij} \mathtt{H}_q(i,j) = n \cdot \mathrm{Trace}(M) + n^2 m_{qq} \tag{37}$$

Hence the difference between (37) and (35), namely $n^2 m_{qq}$, represents the extra distance traveled between two vertices when forced to pass through vertex q, summed over all n^2 pairs of source/destination vertices.

Similarly, one can compute the detour overhead through a node q using the commute

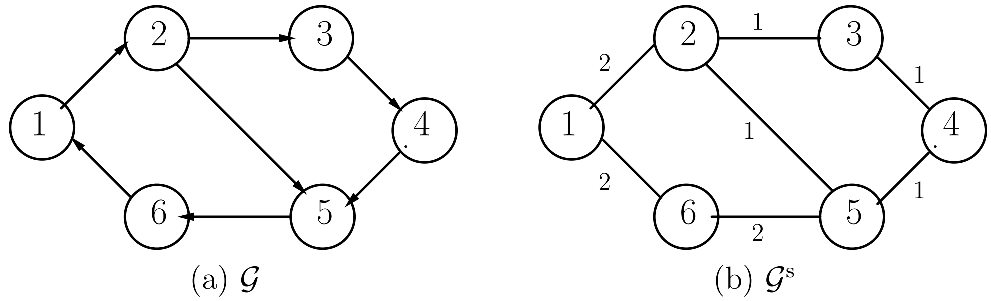


Figure 1: Simple unweighted directed graph $\mathcal{G}$ corresponding to (40) and its corresponding symmetrized weighted undirected graph $\mathcal{G}^{\mathrm{s}}$ derived using (8), (9).

time distances. From equation (36), we get

$$\begin{aligned} \mathtt{C}_q(i,j) &\overset{\text{def}}{=} \mathtt{C}(i,q)+\mathtt{C}(q,j) \\ &= \mathtt{H}(i,q)+\mathtt{H}(q,i)+\mathtt{H}(j,q)+\mathtt{H}(q,j)=\mathtt{H}_q(i,j)+\mathtt{H}_q(j,i) \\ &= \mathtt{H}(i,j)+m_{qq}-m_{iq}-m_{qj}+\mathtt{H}(j,i)+m_{qq}-m_{jq}-m_{qi} \\ &= \mathtt{C}(i,j)+2m_{qq}-m_{iq}-m_{qi}-m_{jq}-m_{qj} \end{aligned} \tag{38}$$

Summing this up for all pairs of sources i and destinations j yields

$$\sum_{ij}\mathtt{C}_q(i,j)=2n\cdot\mathrm{Trace}(M)+2n^2m_{qq} \tag{39}$$

The detour overhead is the same (up to the factor of 2) for the non-metric hitting times and metric commute times for digraphs, much the same way as has been reported for undirected graphs in [30].

7 Examples and Application Scenarios

7.1 An Example

We illustrate some of the results in this work with the help of a simple example. The state transition matrices for the the simple network $\mathcal{G}$ shown in Fig. 1(a) and for the corresponding weighted undirected graph $\mathcal{G}^{\mathrm{s}}$ shown in Fig. 1(b) are

$$P=P(\mathcal{G})=\begin{pmatrix} 0 & 1.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 1.0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad P^{\mathrm{s}}=P(\mathcal{G}^{\mathrm{s}})=\begin{pmatrix} 0 & 0.50 & 0 & 0 & 0 & 0.50 \\ 0.50 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0 & 0.50 & 0 & 0.50 & 0 & 0 \\ 0 & 0 & 0.50 & 0 & 0.50 & 0 \\ 0 & 0.25 & 0 & 0.25 & 0 & 0.50 \\ 0.50 & 0 & 0 & 0 & 0.50 & 0 \end{pmatrix}. \tag{40}$$

The vector of stationary probabilities shared by both $\mathcal{G}$ and $\mathcal{G}^{\mathrm{s}}$ is

$$\boldsymbol{\pi} = \big(0.2,\quad 0.2,\quad 0.1,\quad 0.1,\quad 0.2,\quad 0.2\big)^{\mathrm{T}}.$$

The pseudo-inverses of the Laplacians for these graphs are

$M = L^{+}$ for $\mathcal{G}$ (Fig. 1a)

$$\frac{5}{18}\begin{pmatrix} 9 & 6 & 0 & -6 & -3 & -6 \\ -6 & 9 & 3 & -3 & 0 & -3 \\ -9 & -12 & 18 & 12 & -3 & -6 \\ -3 & -6 & -12 & 18 & 3 & 0 \\ 3 & 0 & -6 & -12 & 9 & 6 \\ 6 & 3 & -3 & -9 & -6 & 9 \end{pmatrix}$$

$M^{\mathrm{s}} = (L^{\mathrm{s}})^{+}$ for $\mathcal{G}^{\mathrm{s}}$ (Fig. 1b)

$$\frac{5}{18}\begin{pmatrix} 19 & 3 & -11 & -13 & -3 & 5 \\ 3 & 15 & -3 & -9 & -3 & -3 \\ -11 & -3 & 31 & 5 & -9 & -13 \\ -13 & -9 & 5 & 31 & -3 & -11 \\ -3 & -3 & -9 & -3 & 15 & 3 \\ 5 & -3 & -13 & -11 & 3 & 19 \end{pmatrix}$$

Following (37), we use the diagonal entry m_{qq} of the pseudo-inverse of the Laplacian as a measure of centrality. Recall, the lower the value of m_{qq}, the *more central* is the node q. For the original graph, nodes 1, 2, 5, 6 are tied as winners in their centrality scores, while in the symmetrized graph, nodes 2, 5 are considered more central compared to nodes 1, 6. We, therefore, see that the centrality ranks are not invariant under the symmetrization process even when the page rank, determined by the vector of stationary probabilities, is the same for both $\mathcal{G}$ and $\mathcal{G}^{\mathrm{s}}$.

The matrices of commute times for the two graphs are (rounded to the digits shown)

C for $\mathcal{G}$ (Fig. 1a)

$$\begin{pmatrix} 0 & 5 & 10 & 10 & 5 & 5 \\ 5 & 0 & 10 & 10 & 5 & 5 \\ 10 & 10 & 0 & 10 & 10 & 10 \\ 10 & 10 & 10 & 0 & 10 & 10 \\ 5 & 5 & 10 & 10 & 0 & 5 \\ 5 & 5 & 10 & 10 & 5 & 0 \end{pmatrix}$$

C for $\mathcal{G}^{\mathrm{s}}$ (Fig. 1b)

$$\begin{pmatrix} 0 & 7.8 & 20.0 & 21.1 & 11.1 & 7.8 \\ 7.8 & 0 & 14.4 & 17.8 & 10.0 & 11.1 \\ 20.0 & 14.4 & 0 & 14.4 & 17.8 & 21.1 \\ 21.1 & 17.8 & 14.4 & 0 & 14.4 & 20.0 \\ 11.1 & 10.0 & 17.8 & 14.4 & 0 & 7.8 \\ 7.8 & 11.1 & 21.1 & 20.0 & 7.8 & 0 \end{pmatrix}$$

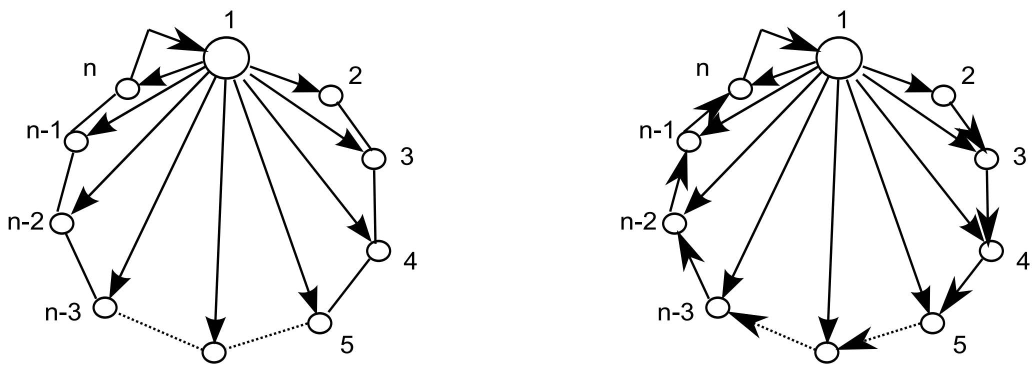
The lower bounds (29) are the same for both these graphs since they depend only on the stationary probabilities, which they share. In this particular case, the lower bounds happen to exactly match the commute times C for $\mathcal{G}$. Hence this example shows the lower bounds can be tight. The upper bounds (31) are

upper bounds (31) for $\mathcal{G}$

$$\begin{pmatrix} 0 & 29.0 & 43.5 & 43.5 & 29.0 & 29.0 \\ 29.0 & 0 & 43.5 & 43.5 & 29.0 & 29.0 \\ 43.5 & 43.5 & 0 & 58.0 & 43.5 & 43.5 \\ 43.5 & 43.5 & 58.0 & 0 & 43.5 & 43.5 \\ 29.0 & 29.0 & 43.5 & 43.5 & 0 & 29.0 \\ 29.0 & 29.0 & 43.5 & 43.5 & 29.0 & 0 \end{pmatrix}$$

upper bounds (31) for $\mathcal{G}^{\mathrm{s}}$

$$\begin{pmatrix} 0 & 53.0 & 79.5 & 79.5 & 53.0 & 53.0 \\ 53.0 & 0 & 79.5 & 79.5 & 53.0 & 53.0 \\ 79.5 & 79.5 & 0 & 106.0 & 79.5 & 79.5 \\ 79.5 & 79.5 & 106.0 & 0 & 79.5 & 79.5 \\ 53.0 & 53.0 & 79.5 & 79.5 & 0 & 53.0 \\ 53.0 & 53.0 & 79.5 & 79.5 & 53.0 & 0 \end{pmatrix}$$



(a) Partial asymmetry (b) Full asymmetry on the chain of low power stations.

Figure 2: A high power broadcaster on a ring.

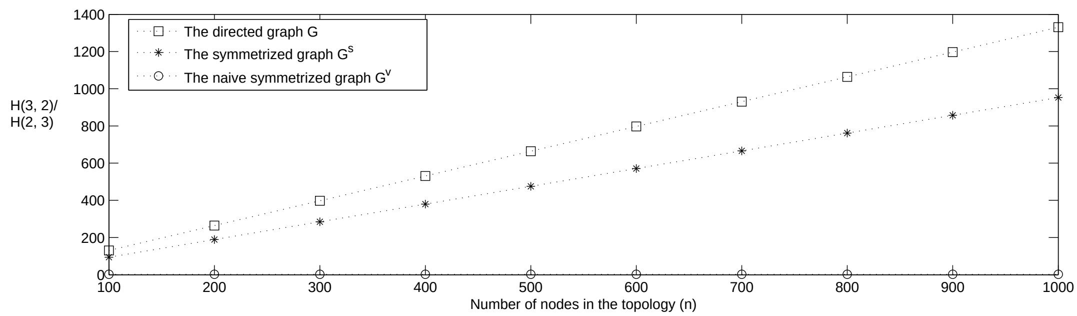


Figure 3: Ratio of Hitting times between nodes 2 and 3 for $\mathcal{G}$, $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{v}}$ for figure 2(a).

Considering that the upper bounds were derived from the summation in (30), it is clear that at least one term in that summation must satisfy $\mathtt{H}(i,j)\pi_j \leq (\mathrm{tr}(Z^{\mathrm{d}}) - 1)/n$, and hence the upper bounds cannot be tight.

7.2 Asymmetric Nature of Wireless Networks

We now use an example motivated by the domain of wireless networks to illustrate how certain graph quantities for the directed graph can be markedly different in the corresponding symmetrized graphs. Wireless networks is one domain where link asymmetry naturally demands modeling of networks as directed graphs. Traditionally, these have been modeled as undirected graphs [6] using various methods of symmetrization for the sake of simplicity. Recently Li & Zhang [23] proposed to treat wireless networks with their asymmetric links *as is* while analyzing the average transmission delays and costs between pairs of nodes in the network. For simplicity, we assign an equal cost to every link in the topology while preserving the link asymmetry. We also confine ourselves to the case of *stateless routing* [6], akin to a random walk over the state space of the wireless devices in the topology, which is relevant to the current work and is applicable to wireless networks due to ease of implementation and maintenance.

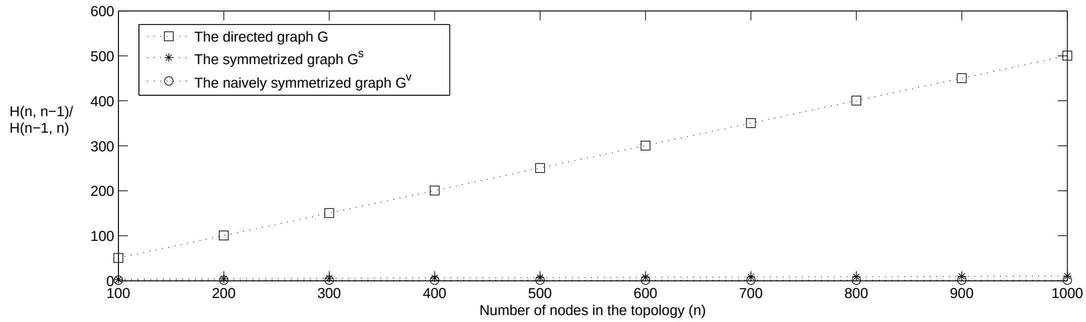


Figure 4: Ratio of Hitting times between nodes $n-1$ and n for $\mathcal{G}$, $\mathcal{G}^{\mathrm{s}}$ and $\mathcal{G}^{\mathrm{v}}$ for figure 2(b).

Consider the topologies in figure 2 with a high power base station, node 1, that can transmit to all the other nodes in the topology through a broadcast. The other low power stations, nodes 2 through n, form a chain-like topology with links to their immediate neighbors. Only node n, henceforth called the terminal node, has a link to the broadcasting base station. It is therefore the egress point of the chain topology. In figure 2 (a), the links connecting the nodes 2 through n to their respective neighbors are symmetric/bi-directional while in 2 (b), each of the low power nodes has an asymmetric link to its neighbor in the clockwise direction. Of course, the connection between nodes 1 and n is bi-directional in both topologies.

We study the hitting times between a pair of nodes in each of the two topologies to observe the effect of approximating a directed graph $\mathcal{G}$ by its symmetrized counterparts $\mathcal{G}^{\mathrm{s}}$ or $\mathcal{G}^{\mathrm{v}}$. The control parameter for the experiment is the number of nodes in the topology which we vary from $n = 100$ to $n = 1000$ in steps of 100.

For the topology in figure 2 (a), we analyze the hitting times between nodes 2 and 3. The numerical values of $\mathtt{H}(2,3)$ and $\mathtt{H}(3,2)$ have been provided in Table 1 for $n = \{100, 500, 1000\}$. While $\mathtt{H}(2,3)$ is constant (≈ 1), $\mathtt{H}(3,2)$ increases consistently with increasing values of n for the original digraph $\mathcal{G}$. This shows that the expected cost of communication from node 3 to node 2 appears to rise linearly with the size of the ring. In figure 3, we plot the ratios $\mathtt{H}(3,2)/\mathtt{H}(2,3)$ for the directed graph $\mathcal{G}$, $\mathtt{H}^{\mathrm{s}}(3,2)/\mathtt{H}^{\mathrm{s}}(2,3)$ the symmetrized graph $\mathcal{G}^{\mathrm{s}}$ and $\mathtt{H}^{\mathrm{v}}(3,2)/\mathtt{H}^{\mathrm{v}}(2,3)$ the naively symmetrized graph $\mathcal{G}^{\mathrm{v}}$. Notice that while the curve monotonically increases with the value of n for $\mathcal{G}$ and $\mathcal{G}^{\mathrm{s}}$, for $\mathcal{G}^{\mathrm{v}}$ it is almost a constant (≈ 2).

Similarly, for the topology in figure 2 (b), we analyze the hitting times between nodes $n-1$ and n, instead. The numerical values of $\mathtt{H}(n-1,n)$ and $\mathtt{H}(n,n-1)$ have been provided in Table 2 for $n = \{100, 500, 1000\}$. Again, $\mathtt{H}(n-1,n)$ is constant (≈ 1) whereas $\mathtt{H}(n,n-1)$ increases consistently with increasing values of n. In figure 4, we plot the ratios $\mathtt{H}(n,n-1)/\mathtt{H}(n-1,n)$ in the directed graph $\mathcal{G}$, $\mathtt{H}^{s}(n,n-1)/\mathtt{H}^{s}(n-1,n)$ the symmetrized graph $\mathcal{G}^{\mathrm{s}}$ and $\mathtt{H}^{\mathrm{v}}(n,n-1)/\mathtt{H}^{\mathrm{v}}(n-1,n)$ the naively symmetrized graph $\mathcal{G}^{\mathrm{v}}$. This time, the curve for $\mathcal{G}$ appears to grow at a much faster rate with growing values of n than for either $\mathcal{G}^{\mathrm{s}}$ or $\mathcal{G}^{\mathrm{v}}$.

From these observations, we see that the hitting times for a digraph and for any of its

Table 1: $\mathtt{H}(2,3)$ and $\mathtt{H}(3,2)$ for the directed graph $\mathcal{G}$ in figure 2 (a) and its symmetrized variants.

	Directed graph $\mathcal{G}$		Symmetrized graph $\mathcal{G}^{\mathrm{s}}$		Naively symmetrized $\mathcal{G}^{\mathrm{v}}$	
n	$\mathtt{H}(2,3)$	$\mathtt{H}(3,2)$	$\mathtt{H}^{\mathrm{s}}(2,3)$	$\mathtt{H}^{\mathrm{s}}(3,2)$	$\mathtt{H}^{\mathrm{v}}(2,3)$	$\mathtt{H}^{\mathrm{v}}(3,2)$
100	1	130.6867	1.3868	130.3000	73.5000	148.5000
500	1	664.0004	1.3963	663.6077	373.5000	748.5000
1000	1	1330.6686	1.3964	1330.2722	748.5000	1498.5000

Table 2: $\mathtt{H}(n-1,n)$ and $\mathtt{H}(n,n-1)$ for the directed graph $\mathcal{G}$ in figure 2 (b) and its symmetrized variants.

	Directed graph $\mathcal{G}$		Symmetrized graph $\mathcal{G}^{\mathrm{s}}$		Naively symmetrized $\mathcal{G}^{\mathrm{v}}$	
n	$\mathtt{H}(n-1,n)$	$\mathtt{H}(n,n-1)$	$\mathtt{H}^{\mathrm{s}}(n-1,n)$	$\mathtt{H}^{\mathrm{s}}(n,n-1)$	$\mathtt{H}^{\mathrm{v}}(n-1,n)$	$\mathtt{H}^{\mathrm{v}}(n,n-1)$
100	1	50.5204	23.8302	70.3670	94.1911	114.8430
500	1	250.5040	60.3551	421.7567	471.8999	581.7167
1000	1	500.5020	87.9956	884.9298	944.0358	1165.30

symmetrizations may differ markedly, apparently without bound.

8 Conclusion

In this work we studied an asymmetric Laplacian under two different scalings for strongly connected digraphs, the pseudo-inverse of which helps compute important graph properties much the same way as is done in the undirected case. In particular, we developed formulas for the average hitting and commute times which mimic the undirected case, and derived some upper and lower points for these quantities. We derived a specific symmetrization of the digraph which preserves the vertices, edge sets, and stationary probabilities, albeit with altered edge weights, allowing one to exploit the wealth of existing knowledge base for undirected graphs. Finally, we motivated the necessity for computing random walk based quantities directly on the asymmetric structure represented by a directed graph through a case study for a wireless network setup. Through it, we demonstrated how approximating a directed graph by a symmetrized version can lead to large discrepancies even when the resulting undirected graph shares the steady state stationary probabilities with the original directed graph.

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